# Taylor expansion based methods to measure credit risk

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**Abstract**

This paper explores the accuracy of the Taylor expansion based approximations to measure the credit risk loss distribution of the Spanish financial system. We also evaluate the accuracy of the risk allocation under VaR and expected shortfall (ES) criteria obtained with the Pykhtin approximation. Finally, we develop two random recoveries models that can capture the correlation between the default and the final recovery and we use the Taylor expansion ideas to approximate the loss distribution. Our results show that the Pykhtin model is not accurate enough in the case of concentrated portfolios and that the Taylor expansion ideas can be used to approximate the loss distribution of a portfolio under random correlated recoveries in a very satisfactory way. Moreover, this kind of random correlated recoveries models could be easily implemented under the Basel capital charge regulation to improve the credit risk measurement.

**Keywords:** Risk management, approximate methods, multifactor adjustments, random recoveries, VaR, expected shortfall.

**JEL classification:** C15, C63, G21

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# 1 Introduction

This paper uses the Taylor expansion based method in Pykhtin (2004) to obtain an analytical estimation of the credit risk loss distribution of the Spanish financial system. We compare the results of the method with those obtained using Monte Carlo simulations. We also propose an approximate method to calculate the loss distribution of a portfolio under random correlated recoveries. We consider that this kind of approximations can be easily implemented under the Basel capital charge regulation.

In previous papers (see García-Céspedes and Moreno (2014a) and García-Céspedes and Moreno (2014b)) we studied the loss distribution of the Spanish financial system using importance sampling techniques and the saddlepoint methods, respectively. Both methods are subject to random variability due to the Monte Carlo simulation, this noise is more intense in the risk allocation process. One of the objectives of this paper is to analyze the suitability of the Taylor expansion based method to measure the credit risk of a portfolio. In the Taylor expansion based method introduced in Pykhtin (2004), the real loss distribution is approximated using the loss distribution of a similar portfolio plus several adjustment terms. This type of expansion is related with the Taylor expansion of the real loss distribution and the adjustment terms are related with the derivatives of the moments of the portfolio loss.

This paper provides two major contributions to the literature. First, we use the approximation in Pykhtin (2004) to measure the risk of the Spanish financial system portfolio built in García-Céspedes and Moreno (2014a) and we allocate it over the different financial institutions. To allocate the risk we use some of the results in Morone et al. (2012). Second, we develop two simple correlated recoveries models and we use the Taylor expansion ideas to approximate the loss distribution under random correlated recoveries.

According to our results, the Pykhtin model does not perform well in the case of concentrated portfolios, it is not able to capture the sudden jumps in the loss distribution, neither it is able to properly allocate the risk over the counterparties. However we get very satisfactory results when we apply the Taylor expansion ideas to approximate the loss distribution of a portfolio under random and correlated recoveries.

This paper is organized as follows. Sections 2 and 3 introduce, respectively, the Vasicek (1987) and the Pykhtin (2004) models. Section 4 describes the Spanish financial institutions portfolio and obtains its loss distribution and risk allocation applying the Pykhtin (2004) model. In Section 5 we develop two correlated recoveries models and use the Taylor expansion ideas to approximate the loss distribution. Finally, Section 6 summarizes our main results and concludes.

## 2 The Vasicek (1987) model

We will start remembering the credit risk model developed in Vasicek (1987). This model states that the value of a counterparty $j$, $V_j$, is driven by a own macroeconomic normal factor $Y_j$ and an idiosyncratic independent normal term $\xi_j$. The own macroeconomic factor $Y_j$ is the linear combination of some more general macroeconomic independent factors $z_f$. Then we can write

$$\begin{aligned} V_j &= r_j Y_j + \sqrt{1-r_j^2}\,\xi_j = r_j \sum_{f=1}^{k} \alpha_{f,j} z_f + \sqrt{1-r_j^2}\,\xi_j \\ &= \sum_{f=1}^{k} \beta_{f,j} z_f + \sqrt{1-\sum_{f=1}^{k} (\beta_{f,j})^2}\,\xi_j \end{aligned} \tag{1}$$

The client $j$ defaults in his obligations if the assets value $V_j$ falls below a given level $k$. As $V_j \sim N(0,1)$ there exist a direct relation between the default threshold $k$ and the historical average default rates of client $j$, $PD_{j,C}$ (or over the cycle default rate), this is $k = \Phi^{-1}(PD_{j,C})$, where $\Phi^{-1}(\cdot)$ denotes the inverse normal distribution. The total losses $L$ of a portfolio made up of $M$ clients can be obtained adding the individual ones, that is,

$$L = \sum_{j=1}^{M} x_j(Y_j, \xi_j) = \sum_{j=1}^{M} EAD_j LGD_j D_j(Y_j, \xi_j)$$

where $D_j$ is a dummy variable that indicates if the clients defaults, $EAD_j$ is the exposure at default (the amount owed by the client $j$, or by the clients in the subportfolio $j$, in the default moment), and $LGD_j$ is the loss given default (the percentage of the final loss after all the recovery process relative to the exposure at default). For the sake of simplicity we will use the notation $EAD_j LGD_j = g_j$.

Given the specification in (1) and conditional to the macroeconomic factors $Z = \{z_1, z_2, \cdots, z_k\}$, the default probability of the client $j$ is

$$\begin{aligned} p_j(Z) &= Prob(D_j = 1|Z) = Prob(V_j \le k) = Prob\left(\xi_j \le \frac{k - r_j \sum_{f=1}^{k} \alpha_f z_f}{\sqrt{1-r_j^2}}\right) \\ &= \Phi\left(\frac{\Phi^{-1}(PD_{j,C}) - r_j \sum_{f=1}^{k} \alpha_f z_f}{\sqrt{1-r_j^2}}\right) \end{aligned}$$

In a granular portfolio (many identical clients) made up of $M$ different subportfolios, the losses conditional to a macroeconomic scenario are

$$L|z = \sum_{j=1}^{M} g_j \Phi\left(\frac{\Phi^{-1}(PD_{j,C}) - r_j \sum_{f=1}^{k} \alpha_f z_f}{\sqrt{1-r_j^2}}\right) = \sum_{j=1}^{M} g_j p_j(Z)$$

If all the clients in the portfolio are exposed to the same macroeconomic factor, the loss at a given probability level $q$ can be obtained just replacing $z$ by $\Phi^{-1}(q)$ in the previous formula. But in the general case of multi-factor and non-granular portfolios, closed-form expressions are not available and Monte Carlo methods or approximate formulas are needed.[1]

# 3 The Pykhtin (2004) approximate model

Given a certain confidence level $q$, Pykhtin (2004) suggests to estimate the value at risk ($VaR(q)$) and the expected shortfall ($ES(q)$) by approximating the loss distribution of the real portfolio through a Taylor expansion. This author considers a granular and unifactorial portfolio as starting point and then adds some adjustment terms to capture the non-granularity and the multifactoriality of the real portfolio. Following Pykhtin (2004), we consider the random variable $\overline{L}$ and define $L_\epsilon = \overline{L} + \epsilon U$, $\epsilon \in \mathbb{R}$, with $U = L - \overline{L}$.[2] Let $t_q(L)$ denote the percentile $q$ of the random variable $L$. Then, a Taylor expansion of $t_q(L_{\epsilon=1})$ around $\epsilon = 0$ leads to

$$t_q(L) = t_q(L_{\epsilon=1}) = t_q(\overline{L}) + \left.\frac{dt_q(L_\epsilon)}{d\epsilon}\right|_{\epsilon=0} + \frac{1}{2}\left.\frac{d^2 t_q(L_\epsilon)}{d\epsilon^2}\right|_{\epsilon=0} + \cdots \qquad (2)$$

[1]See, for instance, Pykhtin (2004), Glasserman and Li (2005) or Voropaev (2011).
[2]It is straightforward to see that $L_{\epsilon=1} = L$.

Gourieroux et al. (2000) and Martin and Wilde (2002) provide an analytical expression for the previous two derivatives

$$\left.\frac{dt_q(L_\epsilon)}{d\epsilon}\right|_{\epsilon=0} = E\left(U|\overline{L}=t_q\left(\overline{L}\right)\right) \tag{3}$$

$$\left.\frac{d^2t_q(L_\epsilon)}{d\epsilon^2}\right|_{\epsilon=0} = -\left.\frac{1}{f_{\overline{L}}(l)}\frac{d}{dl}\left(f_{\overline{L}}(l)var\left(U|\overline{L}=l\right)\right)\right|_{l=t_q(\overline{L})} \tag{4}$$

where $var()$ stands for the variance of a random variable. Pykhtin (2004) suggests to define $\overline{L}$ as the losses of a granular and unifactorial portfolio that depends on a single macroeconomic factor $\overline{Y}=\sum_{f=1}^k b_f z_f$ where $\sum_{f=1}^k b_f^2 = 1$. Intuitively this unique macroeconomic factor should try to capture most of the influence of the different $z_f$ on the losses of the real portfolio. Then, we have

$$\overline{L}(\overline{Y}) = \sum_{j=1}^M g_j \Phi\left(\frac{\Phi^{-1}(PD_{j,C})-a_j\overline{Y}}{\sqrt{1-a_j^2}}\right) = \sum_{j=1}^M g_j \hat{p}_j(\overline{Y}) \tag{5}$$

The terms $a_j$ are the sensitivities of the clients in the granular portfolio to the single macroeconomic factor $\overline{Y}$.[3]

According to the previous definitions of $\overline{L}$ and $\overline{Y}$, we can rewrite the macroeconomic factor $Y_j$ and the asset value of each client $V_j$ as

$$\begin{aligned} Y_j &= \overline{Y}\sum_{f=1}^k \alpha_{j,f}b_f + \sqrt{1-\left(\sum_{f=1}^k \alpha_{j,f}b_f\right)^2}\gamma_j \\ V_j &= r_j\overline{Y}\sum_{f=1}^k \alpha_{j,f}b_f + \sqrt{1-\left(r_j\sum_{f=1}^k \alpha_{j,f}b_f\right)^2}\psi_j \end{aligned} \tag{6}$$

where the random variables $\gamma_j$ and $\psi_j$ are independent of $\overline{Y}$ but $\gamma_j$ and $\gamma_i$ are correlated between clients, and so are $\psi_j$ and $\psi_i$.

Now we can estimate $E(L|\overline{L})$. Looking at equation (6), we have that

$$E\left(L|\overline{L}\right) = E\left(L|\overline{Y}\right) = \sum_{j=1}^M g_j\Phi\left(\frac{\Phi^{-1}(PD_{j,C})-r_j\overline{Y}\sum_{f=1}^k \alpha_{j,f}b_f}{\sqrt{1-\left(r_j\sum_{f=1}^k \alpha_{j,f}b_f\right)^2}}\right) \tag{7}$$

[3] In the next paragraphs we will show a criterion to set $a_j$ such that the first derivative in equation (2) vanishes.

Comparing equations (5) and (7) we can see that if we define $a_j = r_j \sum_{f=1}^{k} \alpha_{j,f} b_f = \sum_{f=1}^{k} \beta_{j,f} b_f$ then equation (3) equals to zero and the first derivative term in the Taylor expansion vanishes. It is also important to note that, given the previous definition of $a_j$, we can rewrite equation (6) in a much shorter way

$$V_j = a_j \overline{Y} + \sqrt{1 - a_j^2}\, \psi_j \tag{8}$$

Then, $V_j$ conditional to $\overline{Y}$ is distributed as $N\left(a_j \overline{Y}, \sqrt{1 - a_j^2}\right)$. The correlation between $V_j$ and $V_i$ conditional to $\overline{Y}$ will be required later in order to obtain equation (4) and it is equal to that between $\psi_j$ and $\psi_i$. Using equations (1) and (8), we can get

$$E(\psi_j \psi_i) = \frac{\sum_{f=1}^{k} \beta_{f,j} \beta_{f,i} - a_i a_j}{\sqrt{(1 - a_j^2)(1 - a_i^2)}} \tag{9}$$

On the other hand, Pykhtin (2004) suggests to set the coefficients $b_f$ that define the single macroeconomic factor $\overline{Y}$ as

$$b_f(q) = \left. \frac{\sum_{j=1}^{M} \alpha_{j,f} g_j p_j(Y_j)}{\sqrt{\sum_{f=1}^{k} \left( \sum_{j=1}^{M} \alpha_{j,f} g_j p_j(Y_j) \right)^2}} \right|_{Y_j = \Phi^{-1}(q)}$$

This is a weighted sum of the expected losses of every client in the portfolio given that each of the own macroeconomic factors is in its $q$ percentile.

To estimate the second derivative in equation (4) we first note that conditioning to $\overline{L}$ is the same as conditioning to $\overline{Y}$. As $\overline{Y} \sim N(0,1)$, we have $\phi'(y) = -y\phi(y)$. Then, equation (4) becomes

$$\begin{aligned} \left. \frac{d^2 t_q(L_\epsilon)}{d\epsilon^2} \right|_{\epsilon=0} &= \left. -\frac{1}{\phi(\overline{Y})} \frac{d}{d\overline{Y}} \left( \frac{\phi(\overline{Y})}{\overline{L}'(\overline{Y})} var(U|\overline{Y}) \right) \right|_{\overline{Y}=\Phi^{-1}(q)} \\ &= \left. \frac{1}{\overline{L}'(\overline{Y})} \left( -v'(\overline{Y}) + v(\overline{Y}) \left( \overline{Y} + \frac{\overline{L}''(\overline{Y})}{\overline{L}'(\overline{Y})} \right) \right) \right|_{\overline{Y}=\Phi^{-1}(q)} \end{aligned} \tag{10}$$

where $v(\overline{Y}) = var(U|\overline{Y})$. Equation (10) requires the following inputs

$$
\begin{aligned}
\overline{L}'(\overline{Y}) &= -\sum_{j=1}^{M} g_j \frac{a_j}{\sqrt{1-a_j^2}} \phi\left(\frac{\Phi^{-1}(PD_{j,C}) - a_j\overline{Y}}{\sqrt{1-a_j^2}}\right) \\
\overline{L}''(\overline{Y}) &= -\sum_{j=1}^{M} g_j \frac{a_j^2}{1-a_j^2} \frac{\Phi^{-1}(PD_{j,C}) - a_j\overline{Y}}{\sqrt{1-a_j^2}} \phi\left(\frac{\Phi^{-1}(PD_{j,C}) - a_j\overline{Y}}{\sqrt{1-a_j^2}}\right) \\
v(\overline{Y}) &= var(U|\overline{Y}) = var(L|\overline{Y}) = var\left(\sum_{j=1}^{M} g_j D_j \middle| \overline{Y}\right) \\
&= E\left(\left(\sum_{j=1}^{M} g_j D_j\right)^2 \middle| \overline{Y}\right) - \left(E\left(\sum_{j=1}^{M} g_j D_j \middle| \overline{Y}\right)\right)^2 \\
&= \sum_{j=1}^{M}\sum_{i=1}^{M} g_j g_i E\left(D_i D_j | \overline{Y}\right) - \left(E\left(\sum_{j=1}^{M} g_j D_j \middle| \overline{Y}\right)\right)^2 \\
&= \sum_{j=1}^{M}\sum_{i\neq j}^{M} g_j g_i E\left(D_i D_j | \overline{Y}\right) + \sum_{j=1}^{M} g_j^2 E\left(D_j | \overline{Y}\right) - \left(\sum_{j=1}^{M} g_j E(D_j|\overline{Y})\right)^2 \\
&= \sum_{j=1}^{M}\sum_{i\neq j}^{M} g_j g_i \Phi_2\left(\Phi^{-1}(\hat{p}_j(\overline{Y})), \Phi^{-1}(\hat{p}_i(\overline{Y})), \rho_{i,j}\right) + \sum_{j=1}^{M} g_j^2 \hat{p}_j(\overline{Y}) \\
&\quad - \left(\sum_{j=1}^{M} g_j \hat{p}_j(\overline{Y})\right)^2 \qquad (11)
\end{aligned}
$$

where $\Phi_2(\cdot,\cdot,\cdot)$ denotes the bivariate cumulative standard normal distribution and $\rho_{i,j}$ is given by equation (9). We explored different algorithms to implement this bivariate distribution[4] and obtained the best results in terms of accuracy, speed, and possibility to be vectorized using the method proposed in Genz (2004).

Equation (10) also requires to compute the first derivative of $v(\overline{Y})$, $v'(\overline{Y})$. Setting $Z_j(\overline{Y}) = \Phi^{-1}(\hat{p}_j(\overline{Y}))$ we have that $Z_j'(\overline{Y}) = \frac{\hat{p}_j'(\overline{Y})}{\phi(Z_j(\overline{Y}))}$ and some alge-

[4]See Chance and Agca (2003) for a general review of these alternatives or Owen (1956), Vasicek (1996), Genz (2004), and Hull (2011) for a more detailed explanation.

bra leads to

$$\frac{d}{d\overline{Y}}\left[\Phi_2\left(Z_j(\overline{Y}), Z_i(\overline{Y}), \rho_{i,j}\right)\right] = \hat{p}_j'(\overline{Y})\Phi\left(\frac{Z_i(\overline{Y}) - \rho_{i,j}Z_j(\overline{Y})}{\sqrt{1-\rho_{i,j}^2}}\right) + \hat{p}_i'(\overline{Y})\Phi\left(\frac{Z_j(\overline{Y}) - \rho_{i,j}Z_i(\overline{Y})}{\sqrt{1-\rho_{i,j}^2}}\right)$$

Therefore we get that

$$\begin{aligned} v'(\overline{Y}) &= 2\sum_{j=1}^{M}\sum_{i\neq j}^{M} g_j g_i \hat{p}_i'(\overline{Y})\Phi\left(\frac{Z_j(\overline{Y}) - \rho_{i,j}Z_i(\overline{Y})}{\sqrt{1-\rho_{i,j}^2}}\right) + \sum_{j=1}^{M} g_j^2 \hat{p}_j'(\overline{Y}) \\ &\quad -2\sum_{j=1}^{M} g_j \hat{p}_j(\overline{Y}) \sum_{j=1}^{M} g_j \hat{p}_j'(\overline{Y}) \end{aligned} \tag{12}$$

Pykhtin (2004) also obtains an analytical approximation for the expected shortfall (ES) given by

$$\begin{aligned} ES_q(L) &= \frac{1}{1-q}\int_q^1 t_s(L)ds = \frac{1}{1-q}\int_q^1 [t_s(\overline{L}) + \Delta t_s(L)]ds \\ &= ES_q(\overline{L}) + \frac{1}{1-q}\int_q^1 \Delta t_s(L)ds = ES_q(\overline{L}) + \Delta ES_q(L) \end{aligned}$$

where

$$\begin{aligned} ES_q(\overline{L}) &= \frac{1}{1-q}\int_{-\infty}^{\Phi^{-1}(1-q)} \overline{L}(\overline{Y})\phi(\overline{Y})d\overline{Y} \\ &= \frac{1}{1-q}\sum_{j=1}^{M} g_j \Phi_2\left(\Phi^{-1}\left(PD_{j,C}\right), \Phi^{-1}(1-q), a_j\right) \\ \Delta ES_q(L) &= -\frac{1}{2(1-q)}\int_{-\infty}^{\Phi^{-1}(1-q)} \frac{1}{\phi\left(\overline{Y}\right)}\frac{d}{d\overline{Y}}\left(\frac{\phi\left(\overline{Y}\right)}{\overline{L}'\left(\overline{Y}\right)} v\left(\overline{Y}\right)\right)\phi(\overline{Y})d\overline{Y} \\ &= -\left.\frac{v\left(\overline{Y}\right)}{2(1-q)}\frac{\phi\left(\overline{Y}\right)}{\overline{L}'\left(\overline{Y}\right)}\right|_{\overline{Y}=\Phi^{-1}(1-q)} \end{aligned}$$

As it can be observed all the information required to obtain the expected shortfall has been obtained previously to estimate the VaR, this is a big advantage of this method.

Now let us focus on two extreme cases:

1. For a single factor and non-granular portfolio, we have $\rho_{i,j} = 0$ and equations (11)-(12) simplify to

$$\begin{aligned} v\left(\overline{Y}\right) &= \sum_{j=1}^{M}\sum_{i\neq j}^{M} g_j g_i \hat{p}_j\left(\overline{Y}\right)\hat{p}_i\left(\overline{Y}\right) + \sum_{j=1}^{M} g_j^2 \hat{p}_j\left(\overline{Y}\right) - \left(\sum_{j=1}^{M} g_j \hat{p}_j\left(\overline{Y}\right)\right)^2 \\ v'\left(\overline{Y}\right) &= 2\sum_{j=1}^{M}\sum_{i\neq j}^{M} g_j g_i \hat{p}_i'\left(\overline{Y}\right)\hat{p}_j\left(\overline{Y}\right) + \sum_{j=1}^{M} g_j^2 \hat{p}_j'\left(\overline{Y}\right) \\ &\quad -2\sum_{j=1}^{M} g_j \hat{p}_j\left(\overline{Y}\right)\sum_{j=1}^{M} g_j \hat{p}_j'\left(\overline{Y}\right) \end{aligned}$$

2. In the case of multifactorial and granular portfolio, equations (11)-(12) simplify to

$$\begin{aligned} v\left(\overline{Y}\right) &= \sum_{j=1}^{M}\sum_{i=1}^{M} g_j g_i \Phi_2\left(\Phi^{-1}(\hat{p}_j\left(\overline{Y}\right)), \Phi^{-1}(\hat{p}_i\left(\overline{Y}\right)), \rho_{i,j}\right) \\ &\quad -\sum_{j=1}^{M}\sum_{i=1}^{M} g_j g_i \hat{p}_i\left(\overline{Y}\right)\hat{p}_j(\overline{Y}) \\ v'\left(\overline{Y}\right) &= 2\sum_{j=1}^{M}\sum_{i=1}^{M} g_j g_i \hat{p}_i'\left(\overline{Y}\right)\left(\Phi\left(\frac{Z_j\left(\overline{Y}\right) - \rho_{i,j} Z_i\left(\overline{Y}\right)}{\sqrt{1-\rho_{i,j}^2}}\right) - \hat{p}_j\left(\overline{Y}\right)\right) \end{aligned}$$

# 4 Portfolio Results

We study the portfolio of financial institutions covered by the Spanish deposit guaranty fund (FGD) at December, 2010. To obtain the risk measures of the portfolio first we need to have an estimate of the probability of default (PD), exposure at default (EAD), loss given default (LGD) and the macroeconomic factor sensitivity ($\alpha$) of each institution.

We estimate the EAD based on the information on assets and liabilities for the Spanish financial institutions at December 2010. This information can be obtained from the AEB, CECA, and AECR webpages.[5] During year 2010 many mergers took place, therefore we sum all the balance information from the different institutions that belong to the same group.

[5] AEB is the Spanish Bank Association, CECA is the Spanish Saving Bank Association, and AECR is the Spanish Credit Cooperatives Association.

For the PD we use the public credit ratings available at December 2010 and the historical observed default rates reported by the rating agencies (see S&P (2009), Moody's (2009) and Fitch (2009)). Based on those data we infer a default probability for each institution. For those financial institutions with no external rating we assign one notch less than the average rating of the portfolio of financial institutions with external rating.

We extend the LGD results for financial institutions in Bennet (2002) to the period 1986-2009 using the FDIC (deposits guarantee fund in United States) public data of default recoveries.

The macroeconomic factor sensitivity, $\alpha$, is set as the one in the Basel accord. Additionally all the financial institutions in the portfolio are exposed to a single macroeconomic factor, the Spanish factor. This is the case for all the institutions but for BBVA and Santander that are exposed to additional geographies. The exposure of those two institutions to the geographic factors is computed according to their net interest income by geography obtained from the public 2010 annual reports. As we have more than one macroeconomic factor we need a factor correlation matrix, we obtain this matrix using the GDP series of the different countries. Finally we compute orthogonalized factors so that we can apply Pykhtin model.

## 4.1 Portfolio VaR and ES

We have tested the accuracy of the approximate formulas in Pykhtin (2004) using the portfolio of the Spanish financial system at December 2010. The left graph in Figure 1 shows the loss distribution obtained using a simple Monte Carlo method (MC) and the Pykhtin approximation. We show the results of the Pykhtin model when only the first term or the first two terms of the approximation are considered (Pykhtin 1, Pykhtin 2). It can be seen that the Pykhtin method underestimates the probability of high losses for our portfolio. Another issue to be noted is that this method generates an approximation of the percentiles of the loss distribution which is smooth (continuous derivatives) and can not capture the sudden jumps of the portfolio loss distribution.

[INSERT FIGURE 1 AROUND HERE]

The right graph in Figure 1 provides the expected shortfall estimates using the Pykhtin (2004) method (Pykhtin 1, 2) and the exact ones obtained using the Monte Carlo method (MC). The conclusions are similar to those obtained from the loss distribution approximation.

## 4.2 Analytical VaR Contributions

The VaR contribution of a client $i$ can be defined as the derivative of the VaR with respect to the current exposure share of the client $i$, this is

$$CVaR_i = \left.\frac{\partial VaR(w_1, \cdots, w_m)}{\partial w_i}\right|_{w_i=1} \tag{13}$$

where the portfolio loss is defined as $L = \sum_{j=1}^{M} [w_j EAD_j] LGD_j D_j$. Under the Pykhtin approximation approach, we need to obtain the derivative of (2) with respect to $w_i$. This can be done numerically by computing the values $t_q(L, w_1, \cdots, w_i, \cdots, w_M)$ and $t_q(L, w_1, \cdots, w_i + \lambda, \cdots, w_M)$ with $\lambda$ being small enough and using these values to approximate the derivative in equation (13). Unfortunately, this method must be repeated for all the clients as the information used to obtain the derivative of the client $j$ can not be used for another client with the corresponding lack of computational synergies.

Alternatively, we can try to derivate equation (2) analytically. However, this derivative involves those of $a_j(w_1, \cdots, w_i, \cdots, w_M)$ with respect to $w_i$ and this increases the complexity of the analytical derivation. This derivation is considerably simplified if we assume $a_j(w_1, \cdots, w_i, \cdots, w_M)$ to be constant, a (naive) case considered in Morone et al. (2012). We will obtain the terms of the analytical approximation of the VaR contributions under this assumption and will compare the results against a numerical derivation rule and against a Monte Carlo risk allocation rule. The analytical derivation rule has a big computational advantage, namely, most of the terms required in the derivation process have already being obtained for the VaR calculation.

Therefore, the VaR contribution of the client $i$ can be obtained as

$$CVaR_i \approx \left.\frac{\partial t_q(\overline{L})}{\partial w_i}\right|_{w_i=1} + \left.\frac{\partial}{2\partial w_i}\left(\left.\frac{d^2 t_q(L_\epsilon)}{d\epsilon^2}\right|_{\epsilon=0}\right)\right|_{w_i=1} \tag{14}$$

Under the assumption that $a_j(w_1, \cdots, w_i, \cdots, w_M)$ is constant, obtaining the first term of the right-hand side of this equation is straightforward considering equation (5). Looking at (10), the second term in equation (14) is given by

$$\frac{1}{2(\overline{L}'(\overline{Y}))^2} \times \Bigg[-\overline{L}'(\overline{Y})\frac{\partial v'(\overline{Y})}{\partial w_i} + v'(\overline{Y})\frac{\partial \overline{L}'(\overline{Y})}{\partial w_i}$$
$$+ \left(\overline{Y} + \frac{\overline{L}''(\overline{Y})}{\overline{L}'(\overline{Y})}\right)\left(\overline{L}'(\overline{Y})\frac{\partial v(\overline{Y})}{\partial w_i} - v(\overline{Y})\frac{\partial \overline{L}'(\overline{Y})}{\partial w_i}\right)$$
$$+ v(\overline{Y})\overline{L}(\overline{Y})\frac{\partial}{\partial w_i}\left(\overline{Y} + \frac{\overline{L}''(\overline{Y})}{\overline{L}'(\overline{Y})}\right)\Bigg]\Bigg|_{\overline{Y}=\Phi^{-1}(1-q),w_i=1}$$

where

$$\left.\frac{\partial \overline{L}'(\overline{Y})}{\partial w_i}\right|_{w_i=1} = g_i \hat{p}_i'(\overline{Y})$$
$$\left.\frac{\partial \overline{L}''(\overline{Y})}{\partial w_i}\right|_{w_i=1} = g_i \hat{p}_i''(\overline{Y})$$
$$\left.\frac{\partial v(\overline{Y})}{\partial w_i}\right|_{w_i=1} = 2\sum_{j\neq i} g_j g_i \Phi_2\left(\Phi^{-1}(\hat{p}_j(\overline{Y})), \Phi^{-1}(\hat{p}_i(\overline{Y})), \rho_{i,j}\right) + 2g_i^2\hat{p}_i(\overline{Y})$$
$$-2g_i\hat{p}_i(\overline{Y})\sum_{j=1}^{M} g_j\hat{p}_j(\overline{Y})$$
$$\left.\frac{\partial v'(\overline{Y})}{\partial w_i}\right|_{w_i=1} = 2\sum_{j\neq i} g_i g_j \hat{p}_i'(\overline{Y})\Phi\left(\frac{Z_j(\overline{Y}) - \rho_{i,j}Z_i(\overline{Y})}{\sqrt{1-\rho_{i,j}^2}}\right)$$
$$+2\sum_{j\neq i} g_i g_j \hat{p}_j'(\overline{Y})\Phi\left(\frac{Z_i(\overline{Y}) - \rho_{i,j}Z_j(\overline{Y})}{\sqrt{1-\rho_{i,j}^2}}\right)$$
$$+2g_i^2\hat{p}_i'(\overline{Y}) - 2g_i\hat{p}_i(\overline{Y})\sum_{j=1}^{M} g_j\hat{p}_j'(\overline{Y}) - 2g_i\hat{p}_i'(\overline{Y})\sum_{j=1}^{M} g_j\hat{p}_j(\overline{Y})$$

The left graph in Figure 2 shows the loss allocation of the 99.9% probability loss level[6] based on the VaR contribution. We show the results for the risk allocation using Monte Carlo method (MC) and three alternatives in the

[6] 25,645 MM€ and 31,053 MM€ for the Pykhtin approximation and the Monte Carlo method, respectively.

Pykhtin approximation derivatives: analytical derivation formula (Pykthin Anal), and two types of numerical derivatives, one that varies the values of $a_j$ and $b_f$ as $\lambda$ varies on the derivation process (Pykthin Num. Der.) and another one that maintains them constant (Pykthin Num. Der. Const.). It can be seen that the results from the analytical derivation rule coincide with those from the numerical derivation rules. We also observe that the Pykhtin rule based method does not provide accurate results for our portfolio. This is obvious for BBVA and Santander as their contributions should be zero.

[INSERT FIGURE 2 AROUND HERE]

In the case of the expected shortfall we have a similar formula

$$\begin{aligned} CES_i &= \left.\frac{\partial ES(w_1, \cdots, w_m)}{\partial w_i}\right|_{w_i=1} \\ &\approx \frac{g_i}{1-q}\Phi_2\left(\Phi^{-1}(PD_{i,C}), \Phi^{-1}(1-q), a_i\right) \\ &\quad -\frac{\phi(\overline{Y})}{2(1-q)(\overline{L}'(\overline{Y}))^2}\left[\frac{\partial v(\overline{Y})}{\partial w_i}\overline{L}'(\overline{Y}) - \frac{\partial \overline{L}'(\overline{Y})}{\partial w_i}v(\overline{Y})\right] \end{aligned}$$

The results for the expected shortfall based risk allocation can be observed in the right graph in Figure 2. We can see that the results obtained using the Pykhtin approximation are quite close to those provided by the Monte Carlo method.

We finish here our study of the Taylor expansions for the Vasicek (1987) model. An obvious further research is to extend the Pykhtin model for the case of market valuation. In this case, the different clients in a portfolio can migrate to several discrete states (not only to the default one) and, on each state, the loans have a different valuation or loss. Therefore, the estimation of $v(y)$ requires to consider all the possible combinations of states of clients $i$ and $j$. This problem is quadratic in the number of clients and in the number of states. For instance, considering 156 clients and a rating scale with 17 grades, we must estimate 156 x 156 x 17 x 17 bivariate cumulative standard normal probabilities. This makes the utilization of the Pykhtin method less interesting in the case of market valuation. In the next Section we will apply the Taylor expansion ideas to approximate the loss distribution in the case of random recoveries in the Vasicek (1987) model.

# 5 Correlated random LGD

So far we have introduced the Taylor approximation method proposed in Pykhtin (2004) and used it to estimate the loss distribution of the portfolio of Spanish financial institutions and its risk contributions. We believe that this Taylor based approximation can be also used to approximate the case of random correlated recoveries. In the next sections we will introduce a simple model of random LGD and use the Taylor expansion ideas to approximate a more general model.

## 5.1 Simple random LGD model

We have developed a very simple random LGD model based on a single macroeconomic factor $z$ that drives both default and recoveries in a granular portfolio. To keep it simple we assume that the recoveries follow a Bernoulli random variable that takes the values $\{0, 1\}$ depending on the value of the random variable $V_{j,LGD}$. According to this we have that

$$L = \sum_{j=1}^{M} x_j = \sum_{j=1}^{M} g_j \; D_{PD,j}\left(V_{j,PD}\right) \; D_{LGD,j}\left(V_{j,LGD}\right)$$

Each client has a default value function ($V_{j,PD}$) and a LGD value function ($V_{j,LGD}$) given by

$$\begin{aligned} V_{j,PD} &= \rho_{j,PD} z + \epsilon_j \sqrt{1-\rho_{j,PD}^2} \\ V_{j,LGD} &= \rho_{j,LGD} z + \psi_j \sqrt{1-\rho_{j,LGD}^2} \end{aligned}$$

As usual, if $V_{j,PD} < \Phi^{-1}(PD_{j,C})$, the client defaults but now we also have that if $V_{j,LGD} < \Phi^{-1}(LGD_{j,C})$ the LGD is 100%. Conditional to a macroeconomic scenario, the loss distribution of each subportfolio $x_j$ is given by

$$x_j|z = \begin{cases} EAD_j & \text{with prob. } p = \Phi\left(\frac{\Phi^{-1}(PD_{j,C}) - \rho_{PD} z}{\sqrt{1-\rho_{j,PD}^2}}\right) \Phi\left(\frac{\Phi^{-1}(LGD_{j,C}) - \rho_{LGD} z}{\sqrt{1-\rho_{j,LGD}^2}}\right) \\ 0 & \text{with prob. } 1-p \end{cases}$$

As the portfolio is made up of many identical clients, the loss distribution of the whole portfolio conditional to a macroeconomic scenario gets reduced to a single value

$$L|z = \sum_{j=1}^{M} EAD_j \Phi\left(\frac{\Phi^{-1}(PD_{j,C}) - \rho_{PD} z}{\sqrt{1-\rho_{j,PD}^2}}\right) \Phi\left(\frac{\Phi^{-1}(LGD_{j,C}) - \rho_{LGD} z}{\sqrt{1-\rho_{j,LGD}^2}}\right) \tag{15}$$

According to this, the 99.9% probability losses can be obtained just replacing $z$ by $\Phi^{-1}(0.999)$ in equation (15).

The left graph in Figure 3 includes the results of this simplified model, using the Vasicek constant LGD model (Const.) as benchmark.[7] We can observe that under this simplified model the random LGD always increases the risk as $\rho_{j,LGD}$ increases, in fact the case $\rho_{j,LGD} = 0\%$ is equivalent to the Vasicek constant LGD model. Intuitively, the effect of the random LGD gets completely diversified for a granular portfolio with $\rho_{LGD} = 0\%$.

[INSERT FIGURE 3 AROUND HERE]

The right graph of this Figure recovers the constant LGD level required to obtain a given 99.9% probability loss under the simplified random LGD model. In the worst case the average LGD of 40% should be multiplied by 2.5 generating a LGD of 100%. This means that the random LGD model with $\rho_{LGD} = 100\%$ is equivalent to a constant LGD model with LGD of 100%.

The Basel capital accord (see Basel Committee on Bank Supervision (2006)) tries to capture the effect of the random LGD using what it is called the downturn LGD which is a stressed LGD. This simplified model would avoid the introduction of an arbitrary downturn LGD. The parameters $\rho_{j,PD}$ and $\rho_{j,LGD}$ can be calibrated using the historical variability of default and recovery rates. However, one would expect that the random independent LGD could generate some diversification effect that may reduce the risk in some situations.[8] This is not the case in this simplified model because of the single macroeconomic factor assumption. The next Subsection considers a more complex model that allows for risk diversification.

## 5.2 Advanced random LGD model

We extend now the previous model to deal with non-granular portfolios and to consider two different macroeconomic factors, one driving the defaults

[7] We use the parameters $PD_{j,C} = 1\%$, $LGD_{j,C} = 40\%$, $\rho_{j,PD} = \sqrt{24\%}$, and $\rho_{j,LGD}$ varying in $[0,1]$.

[8] The simplified model does not allow for risk diversification once $\rho_{PD}$ and $\rho_{LGD}$ are set.

and another one driving the recoveries. Hence, we have

$$\begin{aligned}
V_{j,PD} &= \rho_{j,PD}\, z_{PD} + \epsilon_j \sqrt{1-\rho_{j,PD}^2} \\
V_{j,LGD} &= \rho_{j,LGD}\, z_{LGD} + \psi_j \sqrt{1-\rho_{j,LGD}^2}
\end{aligned}$$

where the macroeconomic factors are correlated with $corr(z_{PD}, z_{LGD}) = \rho_{z_{PD},z_{LGD}}$. Then, $V_{j,LGD}|Z_{PD} \sim N\left(\rho_{j,LGD}\rho_{z_{PD},z_{LGD}} z_{PD},\ \sqrt{1-\rho_{j,LGD}^2 \rho_{z_{PD},z_{LGD}}^2}\right)$.

To apply the ideas in Pykhtin (2004) we can define $\overline{L}$ as the losses of a constant LGD portfolio with $LGD_{j,C} = LGD_j^*$, that is,

$$\overline{L} = \sum_{j=1}^{M} EAD_j\ LGD_j^*\ \Phi\left(\frac{\Phi^{-1}(PD_{j,C}) - \rho_{j,PD} z_{PD}}{\sqrt{1-\rho_{j,PD}^2}}\right)$$

Then, we have that

$$\begin{aligned}
E(L|\overline{L}) &= E\left(L|z_{PD}\right) \\
&= \sum_{j=1}^{M} EAD_j \left[\Phi\left(\frac{\Phi^{-1}(LGD_{j,C}) - z_{PD}\ \rho_{j,LGD}\ \rho_{z_{PD},z_{LGD}}}{\sqrt{1-\rho_{j,LGD}^2\ \rho_{z_{PD},z_{LGD}}^2}}\right)\right. \\
&\quad \left. \times\ \Phi\left(\frac{\Phi^{-1}(PD_{j,C}) - z_{PD}\ \rho_{j,PD}}{\sqrt{1-\rho_{j,PD}^2}}\right)\right] \\
&= \sum_{j=1}^{M} EAD_j\ \Phi(G_j(z_{PD}))\ p_j(z_{PD}).
\end{aligned}$$

Therefore, if we set $LGD_j^* = \Phi\left(G_j(z_{PD})\right)$, we get that $E\left(L - \overline{L}|\overline{L}\right) = 0$ and equation (3) vanishes. To estimate equations (4) and (10) we have that

$$\begin{aligned}
E(L^2|z_{PD}) &= \sum_{j=1}^{M}\sum_{i\neq j}^{M} EAD_j\ EAD_i\ p_j(z_{PD})\ p_i(z_{PD})\ \Phi_2\left(G_j(z_{PD}), G_i(z_{PD}), \rho_{i,j}\right) \\
&\quad + \sum_{j=1}^{M} EAD_j^2\ \Phi(G_j(z_{PD}))\ p_j(z_{PD})
\end{aligned}$$

with

$$\rho_{i,j} = \frac{\rho_{i,LGD}\ \rho_{j,LGD}\left(1-\rho_{z_{PD},z_{LGD}}^2\right)}{\sqrt{1-\rho_{i,LGD}^2\ \rho_{z_{PD},z_{LGD}}^2}\sqrt{1-\rho_{j,LGD}^2\ \rho_{z_{PD},z_{LGD}}^2}}$$

Then

$$\begin{aligned} v(z_{PD}) &= \sum_{j=1}^{M}\sum_{i\neq j}^{M} EAD_j\; EAD_i\; p_j(z_{PD})\; p_i(z_{PD})\; \Phi_2\left(G_j(z_{PD}), G_i(z_{PD}), \rho_{i,j}\right) \\ &\quad + \sum_{j=1}^{M} EAD_j^2\; \Phi\left(G_j(z_{PD})\right)\; p_j(z_{PD}) \\ &\quad - \left(\sum_{j=1}^{M} EAD_j\; \Phi(G_j(z_{PD}))\; p_j(z_{PD})\right)^2 \qquad (16) \end{aligned}$$

In the case of granular portfolios, equation (16) gets simplified to

$$\begin{aligned} v(z_{PD}) &= \sum_{j=1}^{M}\sum_{i=1}^{M} EAD_j\; EAD_i\; p_j(z_{PD})\; p_i(z_{PD})\; \Phi_2\left(G_j(z_{PD}), G_i(z_{PD}), \rho_{i,j}\right) \\ &\quad - \left(\sum_{j=1}^{M} EAD_j\; \Phi(G_j(z_{PD}))\; p_j(z_{PD})\right)^2 \end{aligned}$$

Straightforward algebra from equation (16) allows us to compute the term $v'(z_{PD})$.[9]

Using expression (16) and its derivative, we can approximate the loss distribution of a given portfolio for different values of the parameter $\rho_{z_{PD},z_{LGD}}$. The left graph in Figure 4 provides the loss distribution of a portfolio made up of 100 identical clients with $PD_{j,C} = 1\%$, $LGD_{j,C} = 40\%$, $\rho_{PD} = \sqrt{0.24}$, $\rho_{LGD} = \sqrt{0.24}$, and $\rho_{z_{PD},z_{LGD}}$ varying in $[0, 1]$. It can be observed that the approximate method generates results that are very close to the exact ones.

[INSERT FIGURE 4 AROUND HERE]

The right graph in Figure 4 provides the loss distribution of the previous portfolio if we had a granular portfolio instead of 100 clients and illustrates the high accuracy of the approximation for this case.

We have tested the approximate method with other alternative LGD

[9]However, our examples are based on a numerical derivative estimation of $v'(z_{PD})$ rather than on the analytical alternative due to the complexity of the formula. Additional results are available upon request.

distributions. If $LGD_j = a_j + b_j z_{LGD} + c_j \psi_j$, we have that

$$
\begin{aligned}
E(LGD_j|z_{pd}) &= a_j + \rho_{z_{PD}z_{LGD}} b_j z_{PD} \\
E(LGD_j^2|z_{pd}) &= a_j^2 + b_j^2(1-\rho^2_{z_{PD}z_{LGD}}) + c_j^2 + 2a_j b_j \rho_{z_{PD}z_{LGD}} z_{PD} \\
E(LGD_j LGD_i|z_{pd}) &= a_i a_j + (a_i b_j + a_j b_i)\rho_{z_{PD}z_{LGD}} z_{PD} \\
&\quad + b_i b_j(1-\rho^2_{z_{PD}z_{LGD}} + \rho^2_{z_{PD}z_{LGD}} z_{PD})
\end{aligned}
$$

As before we set $LGD_j^* = E(LGD_j|z_{pd})$, then we have

$$
\begin{aligned}
E(L|\overline{L}) &= \sum_{j=1}^{M} EAD_j \; E(LGD_j|z_{pd}) \; p_j(z_{PD}) \\
v(z_{PD}) &= \sum_{j=1}^{M}\sum_{i\neq j}^{M} EAD_j \; EAD_i \; p_j(z_{PD}) \; p_i(z_{PD}) \; E(LGD_j LGD_i|z_{pd}) \\
&\quad + \sum_{j=1}^{M} EAD_j^2 \; E(LGD_j^2|z_{pd}) \; p_j(z_{PD}) \\
&\quad - \left(\sum_{j=1}^{M} EAD_j \; E(LGD_j|z_{pd}) \; p_j(z_{PD})\right)^2
\end{aligned}
$$

In Figure 5 we show the result of the approximate method and the Monte Carlo method under Gaussian recoveries.[10] The correlation between the two macroeconomic factors varies between 0 and -1, where the value -1 implies the maximum positive correlation between default and loss given default. It can be observed that the model accuracy is very high.

[INSERT FIGURE 5 AROUND HERE]

Finally, for $LGD_j = e^{a_j + b_j z_{LGD} + c_j \psi_j}$, we get

$$
E\left( (LGD_j)^k \middle| \, z_{pd} \right) = e^{k\mu_1 + \frac{1}{2}(k\sigma_1)^2}, \; k = 1, 2
$$

with

$$
\begin{aligned}
\mu_1 &= a_j + \rho_{z_{PD}z_{LGD}} b_j z_{PD} \\
\sigma_1^2 &= b_j^2(1-\rho^2_{z_{PD}z_{LGD}}) + c_j^2
\end{aligned}
$$

[10] We use $PD_{j,C} = 1\%, \; \rho_{PD} = \sqrt{0.24}, \; a_j = 0.19, \; b_j = 0.08$, and $c_j = 0.11$.

Moreover, we obtain

$$E(LGD_j LGD_i | z_{pd}) = e^{\mu_2 + \frac{1}{2}\sigma_2^2}$$

with

$$\begin{aligned}\mu_2 &= (a_i + a_j) + (b_i + b_j)\rho_{z_{PD}, z_{LGD}} z_{PD} \\ \sigma_2^2 &= c_i^2 + c_j^2 + (1 - \rho^2_{z_{PD}, z_{LGD}})(b_i + b_j)^2\end{aligned}$$

Figure 6 shows the results for the lognormal recoveries[11] and illustrates that the model performance is very similar to that of Gaussian recoveries.

[INSERT FIGURE 6 AROUND HERE]

# 6 Conclusions

This paper has described the Taylor expansion based method introduced in Pykhtin (2004) and applied it to measure the credit risk of the Spanish financial system. We have shown that in the case of concentrated portfolios the approximation does not perform very well and it can underestimate the probability of high losses. We have also shown that the risk allocation based on this method does not generate accurate results for our portfolio. This is the case for the VaR and the ES based risk allocation. Additionally we have evaluated different ways to allocate the risk based on three different derivation rules.

We have also introduced a simple model to consider the effect of correlated random recoveries on the Basel capital requirements formula. The main drawback of this model is that it does not allow for any diversification effect as it assumes that the defaults and the recoveries are driven by the same macroeconomic factor.

Finally we have built a more flexible model that considers correlated random recoveries and we have used the Taylor expansion method to approximate the loss distribution. Our results suggest that this a robust method that generates accurate results and that can handle many different recoveries distributions. We believe that this method can be easily implemented under the Basel capital charge framework in order to consider the correlated random recoveries.

[11] We use $PD_{j,C} = 1\%$, $\rho_{PD} = \sqrt{0.24}$, $a_j = -1.88$, $b_j = 0.4$, and $c_j = 0.51$.

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# Appendix of Figures

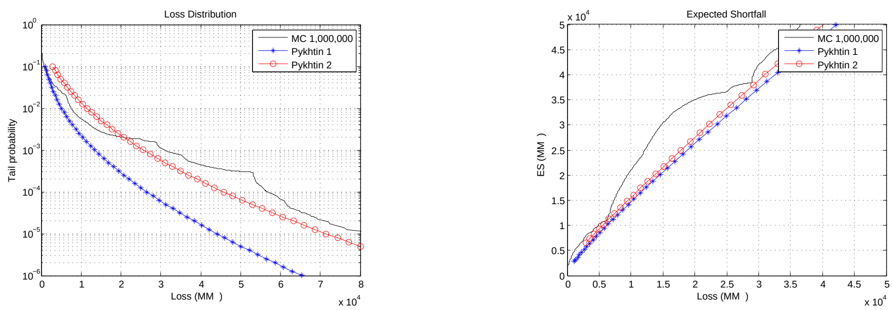


Figure 1: Loss distribution and expected shortfall for the Spanish financial institutions. These results are obtained using a Monte Carlo method with 1,000,000 simulations (MC) and the Pykhtin approximation when only the first term or both terms are considered (Pykhtin 1, Pykhtin 2). The black line indicates the MC results and the blue crosses and the red circles are the Pykhtin 1 and Pykhtin 2 results, respectively. Left and right graphs show the loss distribution and the expected shortfall results, respectively.

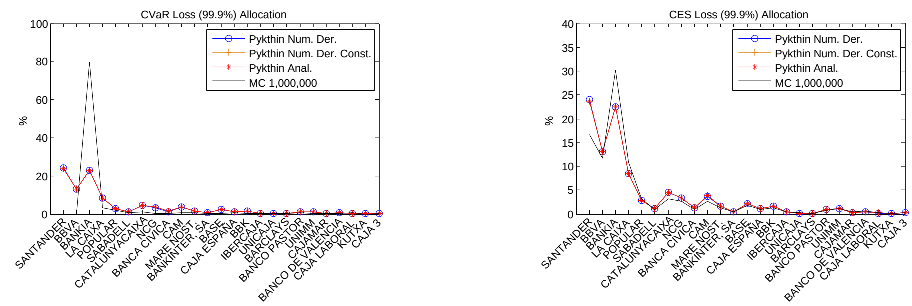


Figure 2: Risk allocation for the Spanish financial institutions. These results are obtained using a Monte Carlo method with 1,000,000 simulations (MC) and the Pykhtin approximation derivatives, using a numerical derivation method and the analytical derivation formula (Pykthin Anal.). We use two types of numerical derivations, one that varies the values of $a_j$ and $b_f$ as $\lambda$ varies on the derivation process (Pykthin Num. Der.) and another one that maintains them constant (Pykthin Num. Der. Const.). Left and right graphs show the loss allocation result based on the VaR and the expected shortfall criteria, respectively.

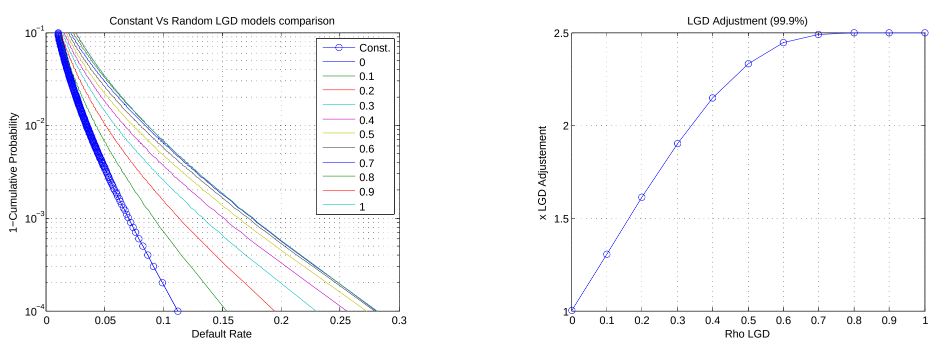


Figure 3: Simplified Random LGD model. The left graph shows the tail loss for the cases of constant LGD (Const.) and for $\rho_{LGD}$ varying in $[0, 1]$. The right graph provides the constant LGD multiplier required to achieve the random LGD 99.9% probability loss as the parameter $\rho_{LGD}$ increases.

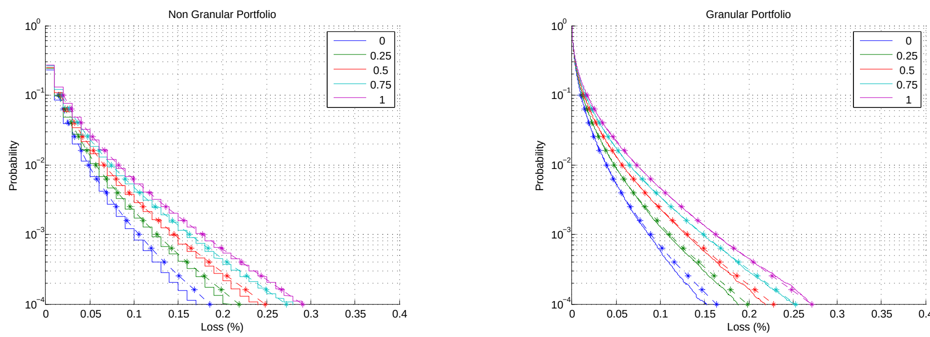


Figure 4: Advanced random LGD model. The left graph shows the results obtained for a portfolio with 100 equal clients for different values of the correlation between the two macroeconomic factors. The continuous lines represent the estimation of the loss distribution obtained using $1,000,000$ Monte Carlo simulations and the dashed lines include the results obtained with the Taylor expansion based approximations. The right graph provides the results for a granular portfolio.

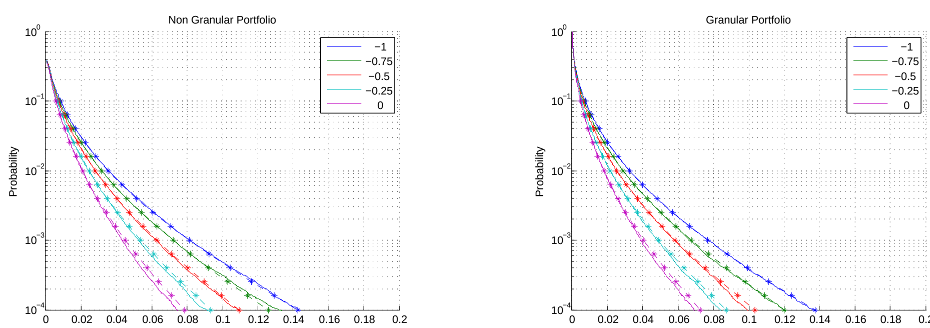


Figure 5: Advanced random LGD model under Normal LGD. The left graph shows the results obtained for a portfolio with 100 equal clients for different values of the correlation between the two macroeconomic factors. The continuous lines represent the estimation of the loss distribution obtained using $1,000,000$ Monte Carlo simulations and the dashed lines include the results obtained with the Taylor expansion based approximations. The right graph provides the results for a granular portfolio.

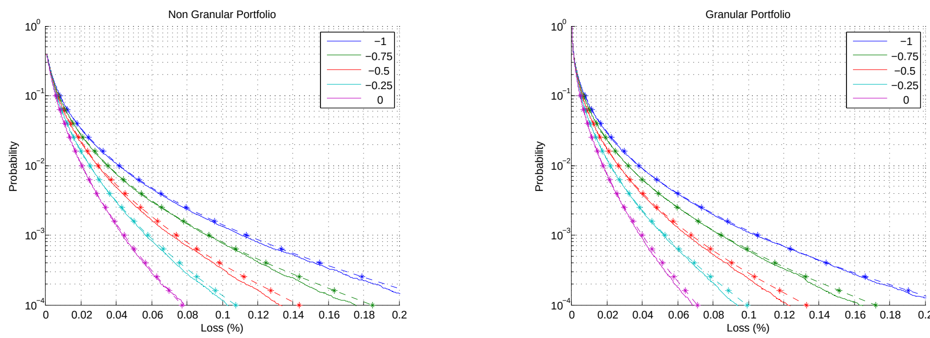


Figure 6: Advanced random LGD model under Lognormal LGD. The left graph shows the results obtained for a portfolio with 100 equal clients for different values of the correlation between the two macroeconomic factors. The continuous lines represent the estimation of the loss distribution obtained using $1,000,000$ Monte Carlo simulations and the dashed lines include the results obtained with the Taylor expansion based approximations. The right graph provides the results for a granular portfolio.