A contingent claim theory of auctions

Federico Platania[a], Pedro Serrano[b,*], Mikel Tapia[c]

[a] *Department of Quantitative Economics, University Complutense of Madrid, 28223 Pozuelo de AlarcÃ³n(Madrid), Spain.*
[b] *Department of Business Administration,University Carlos III, c/Madrid, 126, 28903 Getafe (Madrid), Spain.*
[c] *Department of Business Administration,University Carlos III, c/Madrid, 126, 28903 Getafe (Madrid), Spain.*

Abstract

Limit orders posted in the order book during an auction contain a valuable information about investors' preferences. Based on the connection between the microstructure of the financial markets and the option-pricing literature, this article presents a novel framework where limit orders posted during auctions are understood as European digital options on the trading distribution. Our model describes the order flow dynamics in terms of partial differential equations, and closed-form formulas for the prices of the limit orders are given. Additionally, the state-price density (SPD) of traded prices, the continuous version of Arrow-Debreu prices, is also computed. This density is the endogenous product of trading decisions of limit order investors (Lehmann, 2005). An empirical exercise from actual BME stock auctions is also provided.

Keywords: Auction, Arrow-Debreu asset, State-price density, Digital option, Preopening period.

JEL: G01, G12, G15, G32

*Corresponding author. Tel: (+34)91 624 89 26; Fax: (+34) 91 624 96 07

Email addresses: plataniaf@hotmail.com (Federico Platania), pedrojose.serrano@uc3m.es (Pedro Serrano), mikel.tapia@uc3m.es (Mikel Tapia)

1. Introduction

The role of auctions as mechanisms for constructing efficient prices is of paramount importance in the financial markets. Auctions serve as a reasonable way to fix highly consensual prices for market participants, acting as information aggregation mechanisms. Madhavan (1992) emphasizes the greater price efficiency and reliability of periodic auctions against continuous trading mechanisms, and the relevance of auctions in the price discovery process has received a considerable attention in the literature of financial markets; see, among others, Stoll (2003), Amihud, Mendelson and Pedersen (2005), Madhavan (2000) or O'Hara (1995). This academic interest on auctions is also supported by the marketplace, where auctions are massively used for diverse purposes. For example, main financial markets such as NYSE, NASDAQ or Euronext, employ daily auctions for determining the opening and closing prices (e.g. Biais, Hillion and Spat, 1999); the Deutsche Börse also employs auctions to fix new equilibrium prices at midday, and Bolsas y Mercados Españoles (BME) executes volatility auctions when the stocks trigger predefined price bounds. More recently, the irruption of the computer science and the high frequency traders (HFTs), and their effects on the efficiency and liquidity of transaction prices, has also raised the issue of implementing frequent auctions to improve the quality of transactions and price discovery process (Budish, Cramton and Shim, 2013).

During auctions, agents post orders that disclose their interest to trade. These indicative prices reveal information that could provide valuable insights into the price discovery process. For example, Biais et al. (1999) show that the flow of information of the indicative stock prices increase towards the end of the pre-opening period of the Paris Bourse. The informational content of these prices has usually been eluded in the literature as, for instance, by introducing other alternative assumptions as the presence of a strategic informed traders, or strictly positive probabilities to orders placed during the auction period (see Vives (1995) or Medrano and Vives, 2001). Nevertheless, an interesting proposal to capture the information flows in auctions is that posited by Lehmann (2005) within the context of the limit order book (LOB) continuous trading. Relying on the classical link between the microstructure of financial markets and the contingent claim analysis (e.g. Copeland and Galai, 1983), Lehmann (2005) sketches a method that summarizes all relevant information about investors' risk preferences by means of the implicit state prices embedded in limit orders. His arguments are built on the observation that limit orders are digital options on the probability trade execution, and the value of these options is therefore linked to the preferences of a representative agent.

Market auctions reveal as suitable environments for adapting the contingent claim framework. For example, agents can post, cancel or modify their limit orders, but no trades are executed during the auction period. This particularity converts limit orders in European options, simplifying dramatically the associated pricing formulas. Another reason is that the LOB is partially visible in auctions: investors just observe the (indicative) equilibrium price and volumes. Then, the unexpected arrival of market orders executed against standing limit orders could be disregarded. Taking into consideration these previous arguments, this article presents a model that captures the dynamics of order flow in call auctions. Our approach relies on the mentioned connection between market limit orders and the contingent claim literature: a limit sell order is the payment of one stock in exchange of cash. In the language of options, this payoff is similar to a portfolio composed by a long position in an asset-or-nothing call, and a short position in a cash-or-nothing call (Lehmann, 2005). These options struck at the limit price, and they are written on the distribution of indicative prices. Analogously, the cash flow of a limit buy order is similar to a long cash-or-nothing put, and a short asset-or-nothing put option. Limit orders are just executed at the end of the auction, so these are European style digital options. This distribution of indicative prices at maturity is the distribution of (possible) equilibrium prices, and it is endogenously obtained when combining the orders of bid and ask sides of the LOB. Interestingly, the density of indicative prices represents the state-price density (SPD) of the Arrow-Debreu securities, i.e., the value of 1 dollar whether that contingent state is reached. Since the SPD is a sufficient statistic for pricing, the value of the limit orders could be theoretically obtained (Aït-Sahalia and Lo, 1998).

The main contribution of this article is twofold: first, our setting permits to implicitly obtain an interpretation of limit orders in terms of contingent claims and, subsequently, to extract the implicit state prices from them. Since the time value of money is negligible within the context of microstructure times, the Arrow-Debreu securities are equal to risk-neutral probabilities, and the SPD inferred from limit order prices represents the probability density of indicative prices. This SPD density is endogenously achieved as a combination of the orders placed by investors in the two sides of the LOB. As explained thorough the paper, the order distributions variate because of dissensions about the indicative bid/ask stock price, and the arrival of common information to the marketplace. Along these lines, the model provides a description of the entire orders' distribution of both LOB sides, instead of characterizing the individual behavior of agents. Then, we adopt a novel perspective on moving from individual to aggregate aspects of the market microstructure, where it is not possible to disentangle among others of different agents.

The second contribution is the modeling framework presented here. The assumption of a continuous set of price ticks, where the agents can post their orders at infinitesimal prices, permits us to capture the dynamics of the order flow in terms of partial differential equations (PDE), introducing the mathematics of continuous time in auctions. Considering that limit orders are expressed as options, and options are boundary conditions of PDEs, the option pricing technology of Black and Scholes (1973) and Merton (1973) is directly applicable. The usage of options in microstructure has a long tradition in Finance (see, for instance, Copeland and Galai (1983) or Lehmann, 2005), but its application to the special case of auctions permits to obtain closed-form solutions for the limit orders. Furthermore, some market situations can be understood at light of the option pricing theory. For example, as shown in the paper, the sensibility of option prices to time, the theta (Θ), has a leading role during the random period of the preopening market. Additionally, this option framework is flexible enough to incorporate different market orders as different types of options. In general, our approach generalizes the classical setting of equilibrium prices endogenously formed as deterministic cross of supply and demand functions, translating the classical setting to a probabilistic framework.To the best of our knowledge, this article is the first attempt on describing auctions under a contingent claim setting.

This article is most closely related to Lehmann (2005) and Lehmann (2008). In these articles, the author draws a route map for a contingent claim theory of the LOB, providing an insightful interpretation of limit orders in terms of derivative securities. Our paper benefits from the identification of the Arrow-Debreu prices in order flow states in Lehmann (2005), formalizing the main ideas there within the context of call auctions. Surprisingly enough, our approach converges to that of Lehmann (2005) whether the recent proposals for modifying the structure of LOB are successful. The impact of HFT trading in the LOB market have raised the issue of using continuous auctions as mechanisms for controlling the distortions introduced by HFT traders. Then, Budish et al. (2013) posit a model where frequent batch auctions eliminate the arms race between HFT, reducing the value of tiny speed advantages and transforming competition on speed into competition on price.

This article also belongs to the stream of literature devoted to the analysis of auctions, and their effects of the indicative prices on capturing the price discovery and market efficiency in financial markets. The main conclusion addressed is that correctly-designed call auctions exhibit a positive spillover effect on the dynamic behavior of price formation in continuous market. These results support the idea that auctions lower the execution cost, improve the price discovery process and/or lower the volatility of the stocks. For example, the seminal work of Biais et al. (1999) study the tatonnement process through which markets discover equilibrium prices, showing that indicative prices are informative and that order flow and information revelation do accelerate towards the end of the preopening in the Paris Bourse (today Euronext). Cao, Ghysels and Hatheway (2000) analyze the NASDAQ opening procedure, and Pagano, Peng and Schwartz (2013) explore the impact that NASDAQ calls have on different market quality measures like bid-ask spreads, price volatility, and order routing during the continuous trading. These authors show that the opening and closing calls on NASDAQ reduce the bid-ask spreads and the volatility of stocks. Abad and Pascual (2010) study the role played by a stock-specific circuit breaker which consists of a short-lived call auction triggered by intraday stock-specific price limits. They conclude that market conditions after auctions, proxied by volatility and trading activity, remain unstable till the 90-minute interval following the resumption of the continuous session. Regarding liquidity, as measured by the bid-ask spread, the post-auction levels are only slightly worse than ordinary, and adverse selection costs are reduced to the pre-auction level. Lastly, Comerton-Forde, Ryde and Burridge (2007) analyze the variations in the Hong Kong Exchange opening auctions, concluding that auction design matters on market quality. In particular, these authors observe that the introduction of the opening call auction is followed by a decrease in market quality. Lastly, Biais, BisiÃ¨re and Pouget (2014) test different preopening mechanisms to facilitate the coordination on high equilibrium liquidity and gains from trades. They show that, during nonbinding pre-openings, traders place manipulative orders, reducing the credibility of preplay communication.

Thus, this article characterizes the informational content of limit orders in auctions from a contingent claim perspective. The structure of the article is as follows: Section 2 describes the microstructure of the auctions, and introduces the model. Section 3 shows the relationship of limit orders and options, and presents the state-price density implicit from limit orders. Section 4 presents some additional results of the model, and an empirical exercise is included in Section 5. Finally, Section 6 includes some concluding remarks.

2. A model for the preopening period

This section introduces the model, presenting its most important features and assumptions. For completeness, we firstly provide a brief summary of the microstructure of call auctions. Then, the main constituents of the model – the distribution of orders and order flow – are discussed, to finalize with a formal expression of the equilibrium.

2.1. The microstructure of call auctions

The general procedure of call auctions is similar across markets, and slightly variates from one market to another. Call auctions are time periods where the continuous trading of the market is suspended. During an auction, traders can post market or limit orders that form supply and demand schedules. As orders arrive to the market, an algorithm continuously calculates the indicative (equilibrium) prices and volumes; the indicative price is that which maximizes the indicative volume. At the end of the auction period, trade is conducted at the (single) opening price, based on the supply and demand curves at opening time.

The timing of the auction differs between markets. In general, auction periods use to be the pre-opening or closing times of the market (Biais et al., 1999), and their characteristics differ depending the market. For example, the pre-opening auctions last half-hour, and the opening price is the price that achieves the highest trading volume (e.g. BME). In contrast, the duration of closing auctions are five minutes and, depending on the market, they can close at a pre-specify time or use a random end. Comerton-Forde et al. (2007) provide the comparison of closing auctions in different markets.

2.2. The distribution of orders

Consider a LOB market where investors place their orders on the two possible sides of the book: bid (buy) and ask (sell) sides, respectively. The set of prices, named P, is ordered from the minimum bid to the maximum ask price, and $P \in \mathscr{R}^{+}$. For modeling purposes, allow for the logarithm of prices instead of price ($p = \ln P$). Orders are posted at different prices, and orders belonging to different investors are indistinguishable among them. Then, the distribution of orders is simply the collection of orders at different prices. Figure 1 illustrates this idea: the number of orders (vertical axis) are allocated along the set of prices (horizontal axis). Essentially, this Figure displays the histogram of orders, with bars representing the number of bets placed at a given price at a certain instant. Notice that there are two order distributions, corresponding to the bid (red bars) and ask (blue bars) sides of the LOB, respectively. The higher number of orders in the bid (ask) side of the LOB is placed at USD 10 (15), and orders decrease when separating from this price.

[INSERT FIGURE 1 ABOUT HERE]

Assume now that investors can place orders at any (positive) infinitesimal price, and the price priority is strictly maintained while both size and time priority are ignored (Lehmann, 2005). The solid and dashed lines in Figure 1 plot the continuous distribution abstracted from the bid and ask discrete ones, respectively. This assumption of a continuum of prices leads to a formal expression for the distribution of orders. Let $n_b(p,t)$ be the number of orders in the bid side of the LOB at a given price p and time t. Then, distribution of bid orders is

$$n_b(p,t) = N_b(t) \times g_b(p,t) \tag{1}$$

and a similar reasoning applies for the ask side. Parameter $N_b(t)$ is the total number of orders in the bid side at time t.[1] Additionally, the function $g_b(p,t)$ is the density of bid orders along the set of prices. This density is not the probability of occurrence of an order; function $g_b(p,t)$ refers how the orders are distributed in the bid side of the LOB. For instance, a natural candidate for this density is the Lognormal distribution.

Lastly, the number of bid orders up to a given price is easily obtained by integration of (1) along the set of prices. For example, the total number of orders is computed as,

$$\int_0^\infty n_b(p|t)dp = \int_0^\infty N_b(t)\, g(p|t)dp = N_b(t) \int_0^\infty g(p|t)dp = N_b(t) \tag{2}$$

at a given time t.

2.3. The order flow

A central issue in microstructure consists in describing how the distribution of orders changes over time. This change of orders from one price to other within the same side of

[1] A particular case is when the total number of orders is constant $N_b(t) = N$. Economically, this situation reflects that of a sealed market where, once started the auction, the introduction or cancellation of new orders is banned, but it is allowed to reallocate the orders at different prices.

the LOB is defined here as the order flow. Formally, the order flow is expressed as,

$$\text{order flow} \equiv \frac{\partial n}{\partial t}, \tag{3}$$

i.e., the rate of change of orders with respect to time. The subscripts denoting the bid and ask sides of the LOB have intentionally been dropped, referring to the distribution of orders as n indistinctly – we recover this notation when needed –. The order flow in (3) differs from that of Lehmann (2005), who considers the order flow as the endogenous distribution of indicative prices that emerges when crossing the bid and ask positions. Instead of an endogenous distribution as in Lehmann (2005), this paper names the order flow as the rate of change of orders (exogenously provided) in each side of the LOB.

A simple way of capturing the dynamics of the order flow is by means of partial differential equations (PDEs). Consider the following relationship,

$$\frac{\partial n}{\partial t} = \beta \frac{\partial^2 n}{\partial p^2} \tag{4}$$

with β a positive constant parameter, named diffusion coefficient, that it is analyzed later on. The economic intuition in (4) is obtained when approximating the second order derivative by the difference quotient,

$$\begin{aligned}\frac{\partial n}{\partial t} &\cong \frac{\beta}{\Delta p^2}\left[n(p+\Delta p,t) - 2n(p,t) + n(p-\Delta p,t)\right] \\ &\cong \frac{2\beta}{\Delta p^2}\left[\frac{n(p+\Delta p,t)+n(p-\Delta p,t)}{2} - n(p,t)\right],\end{aligned} \tag{5}$$

and the explanation that follows has been mainly adapted from Farlow (1982). Given the distribution of orders $n(p,t)$ in one side of the LOB (e.g. bid side) at a given time t, if the average of the neighboring number of bids $n(p+\Delta p,t)$ and $n(p-\Delta p,t)$ is lower than $n(p,t)$, then exists a *flow* of orders from price p to their nearby prices $p+\Delta p$ and $p-\Delta p$. In other words, the net flux of orders into p is negative $(\partial^2 n/\partial p^2 < 0)$. Notice how the number of bids $n(p,t)$ in (4) increases $(\partial n/\partial t > 0)$ or decreases $(\partial n/\partial t < 0)$ depending on the sign of the convexity of $n(p,t)$.

Equation (4) can be generalized to include other additional effects affecting the order flow. Assume a general flow function, says J. Rewriting the expression (4) in terms of this function J results,

$$\frac{\partial n}{\partial t} = -\frac{\partial J}{\partial p} \tag{6}$$

where J measures the total flow of orders from one price to nearby prices, and the negative sign has been introduced by convenience as explained in the next section. This total flow J reflects the amount of orders per unit of time and price, and it is also known as flux.

Within the context of our model, the total flow J is composed by the contribution of two orthogonal effects: on the one hand, the dispersion in the distribution of orders due to, for example, the impatience or loss of the consensus with respect to the average price as time goes by, and denoted by J_d. On the other hand, the impact in the orders' distribution of the (arrival of) common information to the market, named by J_f. Mathematically, the total flow is expressed as $J = J_d + J_f$, or equivalently,

$$\frac{\partial J}{\partial p} = \frac{\partial J_d}{\partial p} + \frac{\partial J_f}{\partial p}. \tag{7}$$

We next detail the components of total flow J. The first component, J_d, is the derivative of the distribution of orders with respect to prices,

$$J_d = -\beta \frac{\partial n}{\partial p} \tag{8}$$

and it represents the changes in the distribution of orders with respect to changes in prices. Economically, this term captures the willingness of investors to change their orders as time evolves; abstracting from other external influences, J_d measures the reluctance of investors to disperse from a consensual or average price as time goes by. In the language of the PDEs, J_d measures how the orders diffuse over time, and the negative sign of J_d indicates that the orders flow out that price. The diffusion parameter β controls the speed of this order diffusion, capturing the volatility of this dispersion. Parameter β is defined positive, and it must be statistically estimated.

Figure 2 shows the diffusion effect on the distribution of orders modeled by J_d. At initial time (solid line), bid (ask) orders are distributed around an average log-price 4.6 (5.0). As time goes by (dashed line), the order flow spread out from these average prices to nearby prices, and the distributions of orders flatten.

[INSERT FIGURE 2 ABOUT HERE]

The second component of the order flow is J_f. This function reflect effects such as, for instance, the arrival of (common) new information. These forces are external to the LOB, and their effect result in, for example, a shift in the distribution of orders. Denoting these forces as $F(p,t)$, the flow J_f is defined as,

$$J_f = n(p,t) \times F(p,t). \tag{9}$$

An example of this external stimulus is the constant function $F(p,t) = -\alpha$. This function represents a parallel shift in the distribution of orders, and it models the arrival of common information to the market. Figure 3 shows the external effect modeled by J_f for the case of a constant function. At initial time (solid line), bid (ask) orders are distributed around an average log-price 4.6 (5.0). As time goes by, the entire distributions move to the left (dashed line) as result of, for instance, some new public information arriving to the market.

[INSERT FIGURE 3 ABOUT HERE]

2.4. The equilibrium equation

Once formalized the order flow, it is time to fix the conditions for conserving the number of orders flowing from one price to another over time. This section sets the conditions for the equilibrium of the order flow. Consider the total flow of orders $J(p)$ at price p, and $J(p+dp)$ the flow at a small price increment. Assuming dp small enough, the total flow of orders from one price to another in a small time interval dt,

$$(J(p) - J(p+dp))\,dt = -\frac{\partial J}{\partial p} dp dt \tag{10}$$

and, therefore, the change of orders in a small period of time can be expressed as

$$\frac{\partial n}{\partial t} = -\frac{\partial J}{\partial p}, \tag{11}$$

where this last equation represents the continuity equation, capturing the dynamics of the order flow in a short time interval. Substituting the expression of the total order flow by its components (7) results,

$$\begin{aligned} \frac{\partial n}{\partial t} &= -\left(\frac{\partial J_d}{\partial p} + \frac{\partial J_f}{\partial p}\right) \\ &= \beta \frac{\partial^2 n}{\partial p^2} - \frac{\partial n}{\partial p} F - n \frac{\partial F}{\partial p} \quad . \end{aligned} \tag{12}$$

The model (12) describes the dynamics of the order flow in a context where investors change their orders for discrepancy in the average price and/or the arrival of information to the market. As this model is expressed in terms of a PDE, its solutions are linked to i) the form of the exogenous function $F(p,t)$, ii) the initial conditions (IC), and iii) the boundary conditions.

To provide a general solution to (12), some assumptions should be introduced. Our first assumption concerns the function $F(p,t)$, which is fixed as a constant ($-\alpha$), and the term $\partial F/\partial p$ cancels out. As shown, this constant function reflects a lateral displacement of the entire distribution of orders on the price support. The second assumption refers to the initial condition of (12), and the discussion that follows is focused on the bid side of the LOB – similar arguments are applied to the ask side – . At initial time, all investors share the same opinion about the appropiate bid price. In other words, agents' beliefs about the bid price are unique, and the entire distribution of orders is clustered in *just* one price. Mathematically, this situation is modeled through the Dirac's delta function $\delta(p-p_0)$, with p_0 the (log)bid price at the initial time.

The following proposition provides a solution for the model (12) under the previous assumptions.

Proposition 1. *Let us consider the one-dimensional linear diffusion equation,*

$$\frac{\partial n}{\partial t} = \beta \frac{\partial^2 n}{\partial p^2} + \alpha \frac{\partial n}{\partial p}, \tag{13}$$

where $\beta \in \mathbb{R}^+$ is the diffusion coefficient, and $\alpha \in \mathbb{R}$ is the drift. This PDE has an initial condition of the form

$$n(p,0) = N\,\delta(p-p_0) \tag{14}$$

with N the total number of orders in one side of the LOB, and the boundary conditions are *given as*

$$n(\pm\infty, 0) = \frac{\partial n}{\partial p}(\pm\infty, 0) = 0 \qquad (15)$$

Then, the solution to PDE in (13) is,

$$n(p,t) \quad = \quad \frac{N}{\sqrt{4\pi\beta t}} \exp\left(-\frac{1}{2}\frac{(p+\alpha t - p_0)^2}{2\sigma t}\right) \qquad (16)$$

PROOF. The solution is obtained by applying the Fourier inverse transform, and it is solved in Appendix A. □

Expression (16) shows the distribution of orders over prices and time. Some interesting conclusions arise from this solution. First, orders are lognormally distributed in (the logarithm of) prices, and the distribution of orders expands over time. Second, in absence of exogenous forces, α is equal to zero, and the distribution (16) diffuses over time around an initial price p_0. The speed of this diffusion is controlled by β and, as previously noted, it captures the investors' willingness or reluctance to move their orders as time goes by. Finally, parameter α acts as a drift, modifying the mean of the distribution and displacing the entire distribution of orders. Its effect in the orders' distribution is associated with the impact of an exogenous force; for instance, the arrival of common new information.

3. Equilibrium prices, limit orders and options

This Section completes the model, characterizing the price discovery process. Then, the connection between limit orders and options is also formalized.

3.1. Price discovery

Under the classical setting of supply and demand, the equilibrium price is determined when supply meets demand. Within the context of our model, the price discovery results from the superposition of the bid and ask order distributions. Using the solution in (16), the bid (n_b) and ask (n_a) distributions are,

$$n_b(p,t) = \frac{N_b}{\sqrt{2\pi\sigma_b^2}} \exp\left(-\frac{1}{2}\frac{(p-\mu_b)^2}{\sigma_b^2}\right) \tag{17}$$

$$n_a(p,t) = \frac{N_a}{\sqrt{2\pi\sigma_a^2}} \exp\left(-\frac{1}{2}\frac{(p-\mu_a)^2}{\sigma_a^2}\right) \tag{18}$$

with

$$\mu_a = p_{0,a} - \alpha_a t, \quad \mu_b = p_{0,b} - \alpha_b t \tag{19}$$

$$\sigma_a^2 = 2\beta_a t, \qquad \sigma_b^2 = 2\beta_b t \tag{20}$$

and N_b (N_a) and $p_{0,b}$ ($p_{0,a}$) the total number of orders and log-price at initial time for the bid (ask) side, respectively.

Equations (17) and (18) admit an interpretation in terms of probabilities. When normalized, bid and ask order distributions can be intended as a density of prices. Since a trade happens when bid-ask distributions superpose at a given price and time, the probability of an order execution is the joint probability of both bid and ask orders joint at a certain time. In terms of densities, this is expressed as,

$$\begin{aligned}\rho(p,t) &\equiv \frac{1}{N}\cdot n_a(p,t)\cdot n_b(p,t) \\ &= \frac{1}{N}\frac{1}{2\pi\sigma_a\sigma_b}\exp\left(-\frac{1}{2}\left(\frac{(p-\mu_a)^2}{\sigma_a^2}+\frac{(p-\mu_b)^2}{\sigma_b^2}\right)\right)\end{aligned} \tag{21}$$

with N a normalizing parameter,

$$N = \frac{N_a N_b}{\sqrt{2\pi\left(\sigma_a^2+\sigma_b^2\right)}}\exp\left(-\frac{1}{2}\frac{(\mu_a-\mu_b)^2}{\left(\sigma_a^2+\sigma_b^2\right)}\right). \tag{22}$$

The function $\rho(p,t)$ is the fraction of orders over the total in each time, and it describes the density of cross orders in a given time. This function is important and economically relevant, because it is the joint probability of crossing an bid and ask orders, and it represents the distribution of all possible equilibrium prices. According to (21), the equilibrium price is not unique, and several different equilibrium prices are available at different probabilities. Another important aspect is that $\rho(p,t)$ follows a Gaussian density. Rearranging terms in (21),

$$\rho(p,t) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{1}{2}\frac{(p-\mu)^2}{\sigma^2}\right), \tag{23}$$

and the mean and variance parameters are

$$\mu = \frac{\mu_a\sigma_b^2 + \mu_b\sigma_a^2}{\sigma_a^2+\sigma_b^2}, \quad \sigma^2 = \frac{\sigma_a^2\sigma_b^2}{\sigma_a^2+\sigma_b^2} \quad . \tag{24}$$

Figure 4 illustrates these previous equations. The distribution of auction prices at equilibrium (red line) emerges as bid (blue line) and ask (black line) order distributions interact. The equilibrium variates over time as the order distributions of LOB sides change.

[INSERT FIGURE 4 ABOUT HERE]

From expression (23), the execution log-price (p_e) follows a Gaussian PDF given by the normalized product of the bid and ask log-price PDFs, so $p_e \backsim N(\mu,\sigma)$, where $\mu = p_0 + \alpha_e t$, $\sigma^2 = \beta_e t$, and

$$p_0 = \frac{p_a\beta_b + p_b\beta_a}{\beta_a+\beta_b} \tag{25}$$

$$\alpha_e = -\frac{\alpha_a\beta_b+\alpha_b\beta_a}{\beta_a+\beta_b}, \quad \beta_e = 2\frac{\beta_a\beta_b}{\beta_a+\beta_b} \tag{26}$$

Thus, the equilibrium price is the well-known geometric Brownian motion (GBM),

$$P_e = P_0 \exp\left(\alpha_e t + \sqrt{\beta_e}\cdot B_t\right) \tag{27}$$

with B_t is a Brownian motion $B_t \backsim N(0,\sqrt{t})$. Then, the expected value of the equilibrium price of an auction with bid and ask order distributions in (18) and (17) is,

$$E_T[P_e] = P_0 \exp\left(\alpha_e T + \frac{\beta_e}{2}T\right) \tag{28}$$

A case of particular interest is the price whose transaction probability is higher. Assuming that there is absence of external forces $\alpha_a = \alpha_b = 0$, and the dispersion of price beliefs are equal, $\sigma_a = \sigma_b$, then

$$\mu = \frac{\mu_a \sigma_b^2 + \mu_b \sigma_a^2}{\sigma_a^2 + \sigma_b^2} = \frac{\mu_a + \mu_b}{2}, \tag{29}$$

that is, the mid-point between bid-ask spread.

3.2. Arrow-Debreu securities, options and limit orders

Trading is the endogenous product of bid and ask distributions of the LOB. The distribution (23) results from the trading decisions of market investors, summarizing all the information about the prices in equilibrium. This trading distribution is a density; its area integrates to one, and the area comprised within an infinitesimal price interval p and $p+dp$ represents the probability of one trade at a given price p.

There exists an interesting equivalence between the trading distribution probabilities in (23) and Arrow-Debreu securities. Because interest rates have no role in the microstructure time scales, the time value of money is secondary (Lehmann, 2005). Therefore, (risk-neutral) probabilities and Arrow-Debreu prices are equal. Expression (23) is the state-price density (SPD) of Arrow-Debreu securities, and it summarizes the preferences of a representative agent (Aït-Sahalia and Lo, 1998). The SPD is a fundamental building block for pricing financial securities, and the value of limit orders as options is easily computed with this information.

A limit order can be understood as a portfolio composed by a cash-or-nothing and an asset-or-nothing option, and these options are written on the distribution of possible execution prices (Lehmann, 2005). Let us precise this relationship. In limit sell orders, investors place a bet: if the equilibrium price P_e is equal or higher than the limit price K, they agree to receive K dollars in exchange of one unit of stock. The payoffs of this trade are those of a portfolio composed by a long position in an asset-or-nothing call, and a short position in an cash-or-nothing call. This strategy is shown in Figure 5. If an investor posts a limit sell order for K, her position is worth when the difference between the equilibrium price P_e and the limit price K is positive. In this example, the value of the equilibrium price P_e is 12, the limit sell price is 10, and the option premium is then $12 - 10 = 2$.

[INSERT FIGURE 5 ABOUT HERE]

The price of a limit sell order s is given in the following proposition.

Proposition 2. *The price of a limit sell order s is,*

$$s_0 = c_0^{asset} - c_0^{cash} = P_0 e^{\left(\alpha_e + \frac{\beta_e}{2}\right)T} N(d_2) - KN(d_1) \tag{30}$$

with

$$d_1 = \frac{\ln\left(\frac{P_0}{K}\right) + \alpha_e T}{\sqrt{\beta_e T}}, \quad d_2 = \frac{\ln\left(\frac{P_0}{K}\right) + (\alpha_e + \beta_e)T}{\sqrt{\beta_e T}} \quad . \tag{31}$$

PROOF. The solution is provided in Appendix B. □

The formula (30) consists of two terms: on the one hand, the expected value of the auction's equilibrium price in (28) times the cumulative probability of the SPD density up to limit sell price K; a sort of equilibrium price weighted by the probability of occurrence of limit price K. On the other hand, the limit sell price K times its probability of occurrence. In other words, expression (30) captures the value gain for an investor when placing a limit sell order. This is not a realized gain, because the difference between the equilibrium price and the limit sell price is not due. This formula is in the spirit of Black and Scholes (1973) formula, and it recalls the well-known expression for the call option.

The value of a limit buy order is also expressed in terms of a portfolio of options. A limit buy order consists in the payment of P_e dollars in exchange of one stock unit, if the auction price P_e is equal or lower than the limit price K. These payoffs are those of a portfolio of long position in an cash-or-nothing put option, and a short position in an asset-or-nothing put. Figure 6 shows the payoffs of this strategy. An investor that places a limit buy order for K, her position is worth when the difference between the limit price K and the equilibrium price P_e is positive. In this example, the limit buy price K is 10, the equilibrium price P_e is 8, and the value of the option premium is $10 - 8 = 2$.

[INSERT FIGURE 6 ABOUT HERE]

The value of a limit buy order is provided in the following proposition.

Proposition 3. *The price of a limit buy order b is,*

$$b_0 = p_0^{cash} - p_0^{asset} = KN(-d_1) - P_0 e^{\left(\alpha_e + \frac{\beta_e}{2}\right)T} N(-d_2) \tag{32}$$

with

$$d_1 = \frac{\ln\left(\frac{P_0}{K}\right) + \alpha_e T}{\sqrt{\beta_e T}}, \quad d_2 = \frac{\ln\left(\frac{P_0}{K}\right) + (\alpha_e + \beta_e)\, T}{\sqrt{\beta_e T}} \quad . \tag{33}$$

PROOF. The solution is provided in Appendix B. □

Analogously to the limit sell order, the limit buy order expression in (32) is the composition of i) the limit buy price times its probability of occurrence, and ii) the expected value of the auction's equilibrium price times the cumulative probability of the SPD density up to limit sell price K. As previously mentioned, this formula recalls the put option formula from Black and Scholes (1973) model.

4. Additional results and extensions

4.1. Approximation to the random opening in the LOB markets

All the limit order formulas provided here are applicable to auctions with a fixed time period. However, stock markets use to close the stock auctions with a random time; see, for instance, Biais et al. (1999) for the Paris Bourse, or Abad and Pascual (2007) for the Spanish stock market. In this way, it can be argued than the nature of the limit orders is closer to an American option than an European one. This situation can be circumvented using some well-known results from the option literature.

Consider the preopening call auction of a LOB-driven stock market. The maturity of limit orders placed in this market is T, and a small random time is added to close the auctions – usually no more than 30 seconds – . The values of limit sell (s) and buy (b) orders during the random preopening period are approximated by a first-order Taylor expansion of the option price around time T,

$$s(T + \Delta t) = s(T) + \frac{\partial s}{\partial t}\Delta t + O((\Delta t)^2), \tag{34}$$

$$b(T + \Delta t) = b(T) + \frac{\partial b}{\partial t}\Delta t + O((\Delta t)^2), \tag{35}$$

where Δt denotes a short time period, and $O(\cdot)$ is the remaining approximation term of second order. The derivative of the option value with respect to time represent the option's Θ. The value of the Θ for the limit orders are given in the following proposition,

Proposition 4. *The Θs of the limit sell and buy orders are, respectively,*

$$\begin{aligned}\frac{\partial s_0}{\partial T} &= \frac{\partial c_0^{asset}}{\partial T} - \frac{\partial c_0^{cash}}{\partial T} \\ &= P_0 e^{\left(\alpha_e + \frac{\beta_e}{2}\right)T} \left[\left(\alpha_e + \frac{\beta_e}{2}\right) N(d_2) + \frac{1}{2}\beta_e \left(\frac{\ln\left(\frac{K}{P_0}\right) + (\alpha_e + \beta_e)T}{(\beta_e T)^{3/2}} \right) \rho(d_2) \right] \\ &\quad - \frac{K}{2}\beta_e \left(\frac{\ln\left(\frac{K}{P_0}\right) + \alpha_e T}{(\beta_e T)^{3/2}} \right) \rho(d_1) \end{aligned} \tag{36}$$

and,

$$\begin{aligned}\frac{\partial b_0}{\partial T} &= \frac{\partial p_0^{cash}}{\partial T} - \frac{\partial p_0^{asset}}{\partial T} \\ &= -\frac{K}{2}\beta_e \left(\frac{\ln\left(\frac{K}{P_0}\right) + \alpha_e T}{(\beta_e T)^{3/2}} \right) \rho(d_1) \\ &\quad - P_0 e^{\left(\alpha_e + \frac{\beta_e}{2}\right)T} \left[\left(\alpha_e + \frac{\beta_e}{2}\right) N(-d_2) - \frac{1}{2}\beta_e \left(\frac{\ln\left(\frac{K}{P_0}\right) + (\alpha_e + \beta_e)T}{(\beta_e T)^{3/2}} \right) \rho(d_2) \right] \end{aligned} \tag{37}$$

where $\rho(\cdot)$ denotes the standard normal density function.

PROOF. The solution is obtained from direct application of the formula. □

4.2. Sensitivity to other parameters

The sensitivity of limit orders to other parameters is captured by the option greeks, and they are easily computed from the constituents of limit orders in expressions (32) and (30). Appendix C provides the main results of this section.

5. Empirical Analysis

5.1. Institutional details and data

The Spanish Stock Exchange trading day begins with the opening auction starting at 8:30:00 AM. During 30 minutes, traders can post, cancel or modify orders, but no trades can be executed. Orders that remains in the LOB from previous days also participate. The LOB is partially visible during the auction period, and the indicative equilibrium auction price and the bid/ask volumes subjected to trading are shown. If there is no auction price, the best bid and ask prices are shown, along with the accompanying volumes (and number of orders). The auction period randomly closes from 9:00:00 AM to 9:00:30 AM, to prevent prices from being manipulated. This random end is common for all the market stocks, although the opening auction may be extended 2 additional minutes in special occasions; for example, when there is no indicative price. After the random end, indicative prices convert into equilibrium prices, and orders subject to execution at the fixed auction price are traded. All non-executed orders in the allocation period remain on the order book.

The dataset consists of the preopening auctions for ten stocks quoted int he Spanish stock market and included in the IBEX 35, the Spanish market index. The auction corresponds to March 6th, 2014; this is a representative day of the usual trading in the Spanish market, and it comprises the full information of the LOB for the preopening auctions. The firms are Acciona (ANA), Banco Bilbao Vizcaya Argentaria (BBVA), Bankinter (BKT), Dia (DIA), Grifols (GRF), Iberia (IBE), Inditex (ITX), Repsol (REP), Santander (SAN) and Telefonica (TEF). The dataset has been provided by Bolsas y Mercados Españoles (BME), the holding controlling BME.

5.2. Econometric procedure and results

[WORK IN PROGRESS]

6. Conclusions

Market microstructure literature could benefits from the contingent claim analysis of financial assets. Limit orders are free options offered to market order traders with strike price equal to the limit price (Lehmann, 2005), and the state-price density (SPD) embedded in these limit options could be informative about the preferences of a representative investor; see, for instance, Aït-Sahalia and Lo (1998). Even though this link between the microstructure of financial markets and the option theory has already been noted in the literature (e.g. Copeland and Galai, 1983), some features inherent to the LOB market have limited the potential gains of this approach. For example, limit orders can be executed during the LOB trading, converting limit orders in American options with unknown explicit solutions. Additionally, the unexpected arrival of market orders executed against standing limit orders could break possible equilibrium in the market.

Auctions are suitable environments for applying the contingent claim theory within a microstructure context. For instance, the impact of unexpected market orders is reduced by the partial visibility of the LOB during auctions, and limit orders become European style options – they are executed at the end of the period exclusively. Taking advantage of the situation, this article has developed a microstructure model for stock auctions where limit orders are expressed as a portfolio of options written on the distribution of auction equilibrium prices. In particular, a limit sell order is a long position in an asset-or-nothing call, and a short position in a cash-or-nothing call. Analogously, a limit buy order is long in a cash-or-nothing put and short in an asset-or-nothing put. The model has introduced a continuous-time setting where investors change their orders because of departures about the bid and ask consensus prices and external forces. We have provided a description of the dynamics of the market in terms of partial differential equations, obtaining closed-form expressions for the limit orders.

The main contribution of this article is the expression for the SPD of Arrow-Debreu prices, the building block for pricing financial securities. This density is equivalent to the risk-neutral probability density of equilibrium prices, since interest rates are negligible in the microstructure time context. Moreover, this SPD density emerges as a combination of the trading decisions of investors in the two sides of the LOB.

In conclusion, this article has formalized a novel framework for empirically analyzing the market microstructure of auctions using a contingent claim approach.

Abad, D. and Pascual, R. (2007), 'On the magnet effect of price limits', *European Financial Management Journal* **13**, 833–852.

Abad, D. and Pascual, R. (2010), 'Switching to a temporary call auction in times of high uncertainty', *Journal of Financial Research* **33**, 45–75.

Aït-Sahalia, Y. and Lo, A. W. (1998), 'Nonparametric estimation of state-price densities implicit in financial asset prices', *Journal of Finance* **53**(2), 499–547.

Amihud, Y., Mendelson, H. and Pedersen, L. (2005), 'Foundations and trends in finance', **1**(4), 269–364.

Biais, B., BisiÃ¨re, C. and Pouget, S. (2014), 'Equilibrium discovery and preopening mechanisms in an experimental market', *Management Science* (60).

Biais, B., Hillion, P. and Spat, C. (1999), 'Preopening period in the paris bourse', *Journal of Political Economy* **107**(6), 1218–1248.

Black, F. and Scholes, M. (1973), 'The pricing of options and corporate liabilities', *Journal of Political Economy* **81**(3), 637–654.

Budish, E., Cramton, P. and Shim, J. (2013), The high-frequency trading arms race: frequent batch auctions as a market design response. Working paper.

Cao, C., Ghysels, E. and Hatheway, F. (2000), 'Price discovery without trading: Evidence from the nasdaq preopening.', *The Journal of Finance* (55), 1339–1365.

Comerton-Forde, C., Ryde, J. and Burridge, H. (2007), 'Not all call auctions are created equal: evidence from hong kong', *Review of Quantitative Finance and Accounting* **26**(4), 395–413.

Copeland, T. and Galai, D. (1983), 'Information effects on the bid-ask spread', *Journal of Finance* **38**, 1457–1469.

Farlow, S. J. (1982), *Partial Differential Equations for Scientists and Engineers*, John Wiley and Sons, New York.

Lehmann, B. N. (2005), 'Notes for a contingent claims theory of limit order markets'. Working paper.

Lehmann, B. N. (2008), 'Arbitrage-free limit order books and the pricing of order flow risk'. NBER Working Paper No. 13848.

Madhavan, A. (1992), 'Trading mechanisms in securities markets', *The Journal of Finance* **47**(2), 607–641.

Madhavan, A. (2000), 'Market microstructure: a survey', *Journal of Financial Markets* **3**(3), 205–258.

Medrano, L. A. and Vives, X. (2001), 'Strategic behavior and price discovery', *Rand Journal of Economics* **32**, 221–248.

Merton, R. C. (1973), 'Theory of rational option pricing', *Bell Journal of Economics and Management Science (The RAND Corporation)* **4**(1), 141–183.

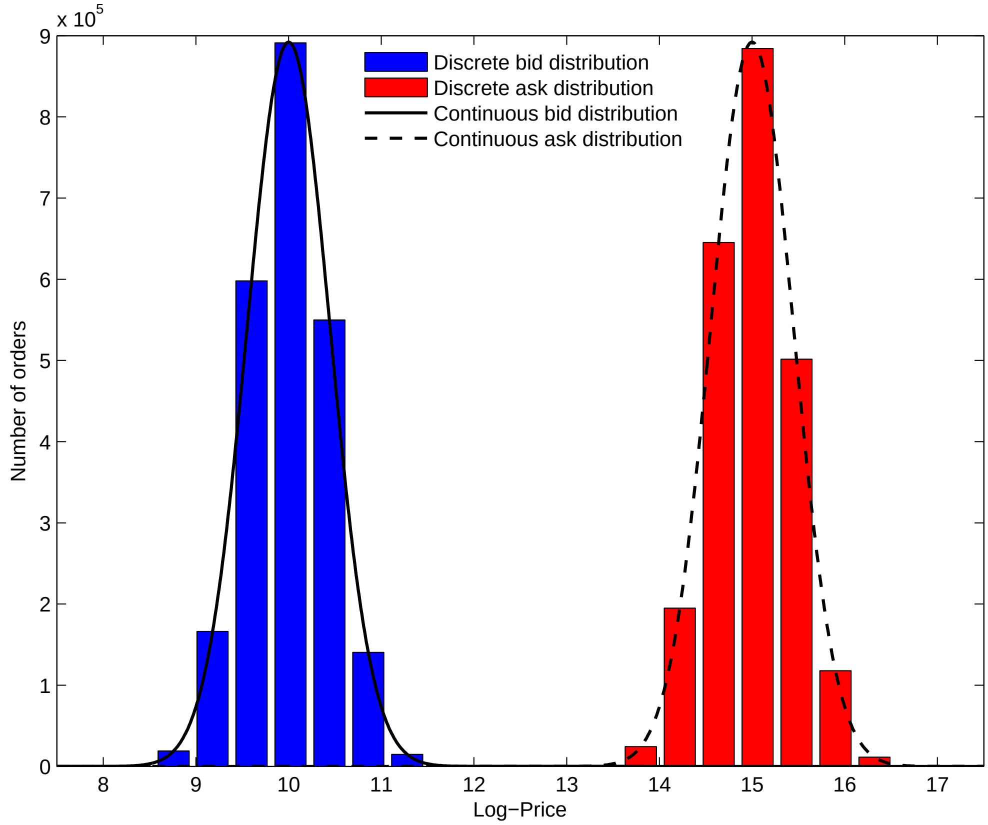
O'Hara, M. (1995), *Market Microstructure Theory*, Blackwell, Cambridge, MA.

Pagano, M., Peng, L. and Schwartz, R. (2013), 'A call auction's impact on price formation and order routing: evidence from the nasdaq stock market.', *Journal of Financial Markets* **16**, 331–361.

Stoll, H. (2003), Market microstructure, *in* Constantinides, Harris and Stulz, eds, 'Handbook of the Economics of Finance', Elsevier.

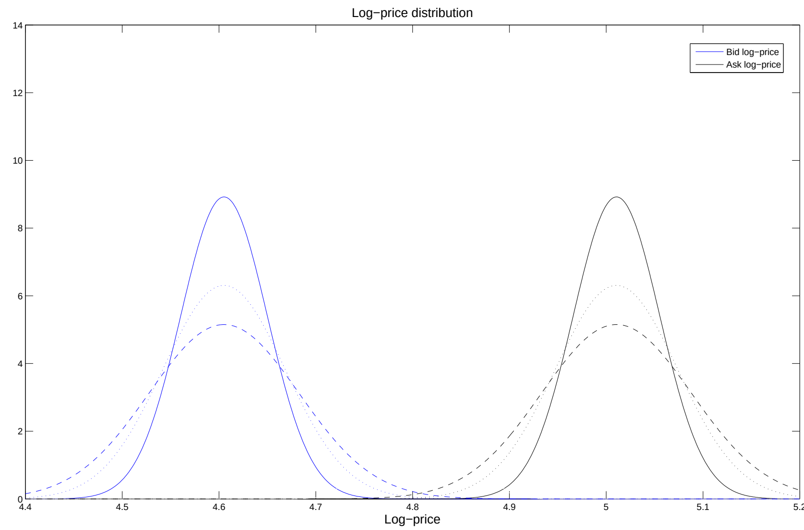
Vives, X. (1995), 'Short term investment and the informational efficiency of the market', *Review of Financial Studies* **8**, 125–160.

Figure 1: The distribution of orders



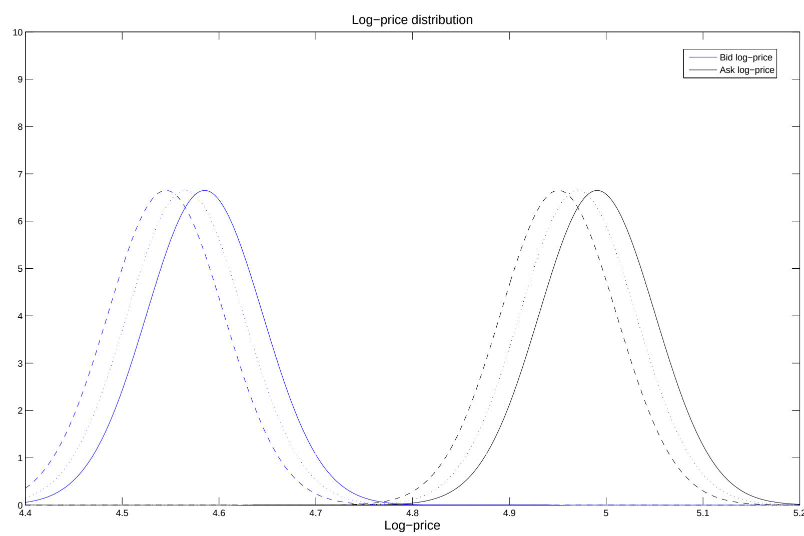
This figure shows the distribution of bid (blue bars) and ask (red bars) orders at different log-prices. Solid and dashed lines depict the continuous representation of bid and ask order distributions, respectively.

Figure 2: Diffusion effect on order flow



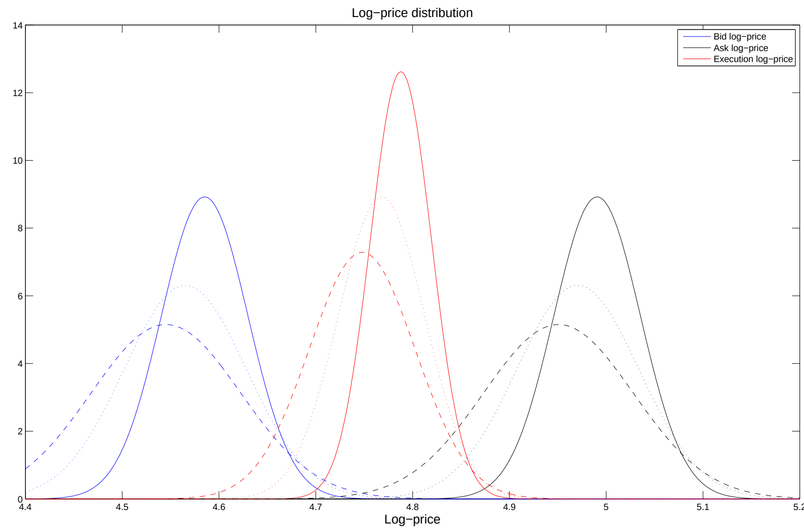
This Figure displays the order flow considering the diffusion effect exclusively. Parameters are $\beta_b = \beta_a = 0.001$. The solid line corresponds to the initial time.

Figure 3: External forces effect on order flow



This Figure depicts the order flow under external forces. Parameters are $\alpha_b = \alpha_a = 0.02$. The solid line corresponds to the initial time.

Figure 4: Distribution of equilibrium prices



This Figure displays the distribution of auction prices at equilibrium (red line). The bid and ask distributions correspond to the blue and black lines, respectively. Parameters are $\alpha_b = \alpha_a = 0.02$ and $\beta_b = \beta_a = 0.001$. The solid lines correspond to the distributions at initial time, and the dashed lines are the distributions over time.

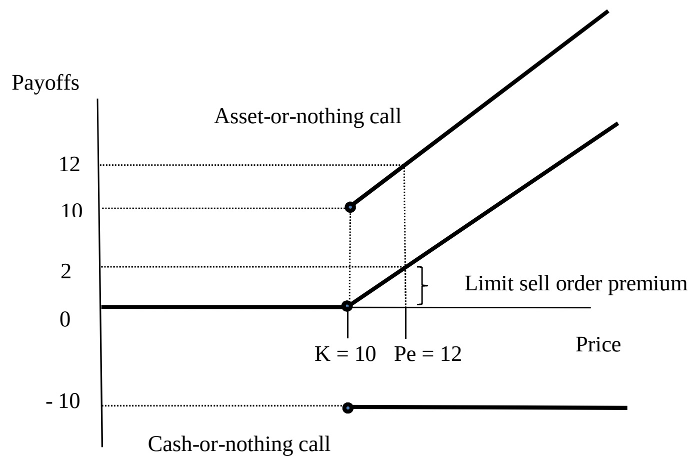


Figure 5: Limit sell order payoffs

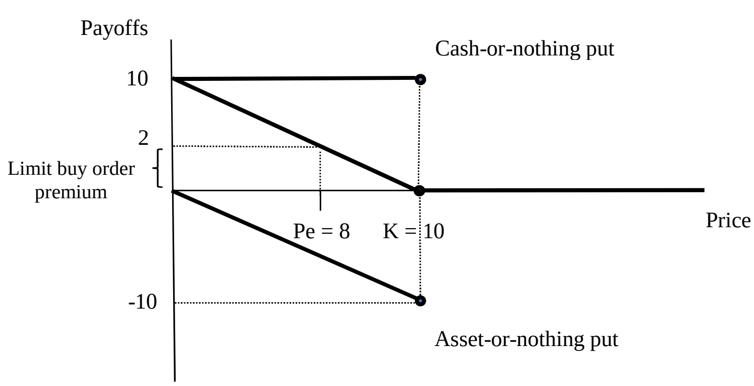


Figure 6: Limit buy order payoffs

Appendix A. Solution for the EDP

Let us consider the one-dimensional linear diffusion equation,

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial u}{\partial x} + \beta \frac{\partial^2 u}{\partial x^2} \tag{A.1}$$

where $\alpha \in \mathbb{R}$ represents the drift coefficient, and $\beta \in \mathbb{R}^+$ the diffusion coefficient. This PDE has an initial condition of the form

$$u(x,0) = f(x) \tag{A.2}$$

and the boundary conditions are given as

$$u(\pm\infty,0) = \frac{\partial u}{\partial x}(\pm\infty,0) = 0 \tag{A.3}$$

Note that when the drift coefficient is zero, the diffusion equation is the heat equation. Otherwise, the diffusion equation is a special case of the Fokker-Planck equation. In order to solve the diffusion equation, consider the following Fourier transform and its inverse

$$\hat{u}(\omega,t) = \mathscr{F}\{u(x,t)\} \tag{A.4}$$

$$u(x,t) = \mathscr{F}^{-1}\{\hat{u}(w,t)\} \tag{A.5}$$

Using the properties of the Fourier transform we get that

$$\frac{\partial u}{\partial x}(x,t) = i\omega \cdot u(x,t) \tag{A.6}$$

$$\frac{\partial^2 u}{\partial x^2}(x,t) = -\omega^2 \cdot u(x,t) \tag{A.7}$$

Hence, applying the Fourier transform to both sides of equation 2 yields

$$\frac{\partial \hat{u}}{\partial t} = (i\omega\alpha - \omega^2\beta)\hat{u} \tag{A.8}$$

where the initial condition is given by $\hat{u}(\omega,0) = \hat{f}(\omega)$. Then, the solution to this initial value problem is of the form

$$\hat{u}(\omega,t) = \hat{f}(\omega)e^{(i\omega\alpha-\omega^2\beta)t} = \hat{f}(\omega)\cdot\hat{g}(\omega) \tag{A.9}$$

where

$$\mathscr{F}^{-1}\{\hat{g}(\omega)\} = \frac{1}{\sqrt{4\pi\beta t}} e^{-\frac{1}{2}\frac{(x+\alpha t)^2}{2\beta t}} \tag{A.10}$$

Applying the inverse Fourier transform to equation A.9, considering the convolution property, and equation A.10, the solution to the diffusion equation is given by

$$\begin{aligned}
\mathscr{F}^{-1}\{\hat{u}(w,t)\} &= \mathscr{F}^{-1}\{\hat{f}(\omega)\cdot\hat{g}(\omega)\} && \text{(A.11)}\\
&= \mathscr{F}^{-1}\{\hat{f}(\omega)\} * \mathscr{F}^{-1}\{\hat{g}(\omega)\} && \text{(A.12)}\\
&= \frac{1}{\sqrt{4\pi\beta t}} \int_{-\infty}^{\infty} f(y) e^{-\frac{1}{2}\frac{(x+\alpha t-y)^2}{2\beta t}} dy && \text{(A.13)}
\end{aligned}$$

The problem has been reduced to an integral, where we have to choose the initial condition $f(x)$. We choose the delta function as the initial condition to the diffusion equation, that is $\delta(x-x_0)$, then

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi\beta t}} \int_{-\infty}^{\infty} \delta(y-x_0) e^{-\frac{1}{2}\frac{(x+\alpha t-y)^2}{2\beta t}} dy && \text{(A.14)}\\
&= \frac{1}{\sqrt{4\pi\beta t}} e^{-\frac{1}{2}\frac{(x+\alpha t-x_0)^2}{2\beta t}} && \text{(A.15)}
\end{aligned}$$

Appendix B. Option pricing

The pricing of the securities composing limit orders is easily computed with the information of the SPD. For example, the cash-or-nothing option is a contract that pays some fixed amount of cash if the option expires in the money, then

$$c_T = \mathscr{H}(P_e - K) \tag{B.1}$$

where $\mathscr{H}(x)$ represents the Heaviside function. Hence, the call price yields

$$\begin{aligned} c_0 &= E\left[\mathscr{H}(P_e - K)\right] \\ &= \int_{-\infty}^{\infty} \mathscr{H}(P_e - K)\, f_t(P_e) dB_T \\ &= \int_{-\infty}^{\infty} \mathscr{H}(P_e - K)\, \frac{e^{-\frac{1}{2}\frac{B_T^2}{T}}}{\sqrt{2\pi T}} dB_T \\ &= N(d_1) \end{aligned} \tag{B.2}$$

where $d_1 = \left(\ln\left(\frac{P_0}{K}\right) + \alpha_e T\right)\sqrt{\beta_e T}$, and the risk-free rate discount has been intentionally skipped as previously mentioned. Analogously, the price of a cash-or-nothing put option is

$$\begin{aligned} p_0 &= E\left[\mathscr{H}(K - P_e)\right] = \int_{-\infty}^{\infty} \mathscr{H}(K - P_e)\, \frac{e^{-\frac{1}{2}\frac{B_T^2}{T}}}{\sqrt{2\pi T}} dB_T \\ &= N(-d_1) \end{aligned} \tag{B.3}$$

An asset-or-nothing call option pays the value of the underlying if the option expires in the money. Then, the payoff of an asset or nothing call option at maturity is given by,

$$c_T = \mathscr{H}(P_e - K)\, P_e \tag{B.4}$$

where again, $\mathscr{H}(x)$ represents the Heaviside function. Hence, the call price yields

$$\begin{aligned} c_0 &= E\left[\mathscr{H}(P_e - K)\, P_e\right] \\ &= \int_{-\infty}^{\infty} \mathscr{H}(P_e - K)\, P_0 e^{\alpha_e T + \sqrt{\beta_e}\, B_T}\, \frac{e^{-\frac{1}{2}\frac{B_T^2}{T}}}{\sqrt{2\pi T}} dB_T \\ &= P_0 e^{\left(\alpha_e + \frac{\beta_e}{2}\right)T}\, N(d_2) \end{aligned} \tag{B.5}$$

where $d_2 = \left(\ln\left(\frac{P_0}{K}\right) + (\alpha_e + \beta_e)\, T\right) / \sqrt{\beta_e T}$.

Analogously, the payoff of an asset-or-nothing put option is given as

$$\begin{aligned} p_0 &= E\left[\mathscr{H}(K - P_e)\, P_e\right] \\ &= \int_{-\infty}^{\infty} \mathscr{H}(K - P_e)\, P_0 e^{\alpha_e T + \sqrt{\beta_e}\, B_T}\; \frac{e^{-\frac{1}{2}\frac{B_T^2}{T}}}{\sqrt{2\pi T}} dB_T \\ &= P_0 e^{\left(\alpha_e + \frac{\beta_e}{2}\right)T}\, N(-d_2) \end{aligned} \tag{B.6}$$

Appendix C. Limit order option greeks

This appendix provides the sensitivity of limit order options to different variables. Here are the list of limit order sensititivities,

$$
\begin{aligned}
\Delta s_0 &= \frac{\partial c_0^{asset}}{\partial P_0} - \frac{\partial c_0^{cash}}{\partial P_0} \\
&= e^{\left(\alpha_e + \frac{\beta_e}{2}\right)T}\left(N(d_2) + \frac{\rho(d_2)}{\sqrt{\beta_e T}}\right) - \frac{\rho(d_1)}{P_0\sqrt{\beta_e T}} \qquad (C.1)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial s_0}{\partial K} &= \frac{\partial c_0^{asset}}{\partial K} - \frac{\partial c_0^{cash}}{\partial K} \\
&= -\frac{P_0 e^{\left(\alpha_e + \frac{\beta_e}{2}\right)T}}{K\sqrt{\beta_e T}}\,\rho(d_2) + \frac{\rho(d_1)}{K\sqrt{\beta_e T}} \qquad (C.2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial s_0}{\partial \alpha_e} &= \frac{\partial c_0^{asset}}{\partial \alpha_e} - \frac{\partial c_0^{cash}}{\partial \alpha_e} \\
&= T P_0 e^{\left(\alpha_e + \frac{\beta_e}{2}\right)T}\left(N(d_2) + \frac{\rho(d_2)}{\sqrt{\beta_e T}}\right) + T\,\frac{\rho(d_1)}{\sqrt{\beta_e T}} \qquad (C.3)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial s_0}{\partial \beta_e} &= \frac{\partial c_0^{asset}}{\partial \beta_e} - \frac{\partial c_0^{cash}}{\partial \beta_e} \\
&= \frac{T}{2} P_0 e^{\left(\alpha_e + \frac{\beta_e}{2}\right)T}\left[N(d_2) + \rho(d_2)\left(\frac{\ln\left(\frac{K}{P_0}\right) - (\alpha_e - \beta_e)T}{(\beta_e T)^{3/2}}\right)\right] \\
&+ \frac{T}{2}\left(\frac{\ln\left(\frac{P_0}{K}\right) - \alpha_e T}{(\beta_e T)^{3/2}}\right)\rho(d_1) \qquad (C.4)
\end{aligned}
$$

where $d_1 = \left(\ln\left(\frac{P_0}{K}\right) + \alpha_e T\right)/\sqrt{\beta_e T}$, and $d_2 = \left(\ln\left(\frac{P_0}{K}\right) + (\alpha_e + \beta_e)T\right)/\sqrt{\beta_e T}$, and $\rho(\cdot)$ denotes the standard normal density function.

The limit buy order sensitivities are listed here,

$$
\begin{aligned}
\Delta p_0 &= \frac{\partial p_0^{cash}}{\partial P_0} - \frac{\partial p_0^{asset}}{\partial P_0} \\
&= -\frac{\rho(d_1)}{P_0\sqrt{\beta_e T}} - e^{\left(\alpha_e + \frac{\beta_e}{2}\right)T}\left(N(-d_2) - \frac{\rho(d_2)}{\sqrt{\beta_e T}}\right) \qquad (C.5)
\end{aligned}
$$

$$\frac{\partial b_0}{\partial K} = \frac{\partial p_0^{cash}}{\partial K} - \frac{\partial p_0^{asset}}{\partial K}$$

$$= \frac{\rho(d_1)}{K\sqrt{\beta_e T}} - \frac{P_0 e^{\left(\alpha_e + \frac{\beta_e}{2}\right)T}}{K\sqrt{\beta_e T}}\,\rho(d_2) \tag{C.6}$$

$$\frac{\partial b_0}{\partial \alpha_e} = \frac{\partial p_0^{cash}}{\partial \alpha_e} - \frac{\partial p_0^{asset}}{\partial \alpha_e}$$

$$= T\,\frac{\rho(d_1)}{\sqrt{\beta_e T}} - T P_0 e^{\left(\alpha_e + \frac{\beta_e}{2}\right)T}\left(N(-d_2) - \frac{\rho(d_2)}{\sqrt{\beta_e T}}\right) \tag{C.7}$$

$$\frac{\partial b_0}{\partial \beta_e} = \frac{\partial p_0^{cash}}{\partial \beta_e} - \frac{\partial p_0^{asset}}{\partial \beta_e}$$

$$= \frac{T}{2}\left(\frac{\ln\left(\frac{P_0}{K}\right) - \alpha_e T}{(\beta_e T)^{3/2}}\right)\rho(d_1)$$

$$- \frac{T}{2} P_0 e^{\left(\alpha_e + \frac{\beta_e}{2}\right)T}\left[N(-d_2) - \rho(d_2)\left(\frac{\ln\left(\frac{K}{P_0}\right) - (\alpha_e - \beta_e)T}{(\beta_e T)^{3/2}}\right)\right] \tag{C.8}$$

where $d_1 = \left(\ln\left(\frac{P_0}{K}\right) + \alpha_e T\right) / \sqrt{\beta_e T}$, and $d_2 = \left(\ln\left(\frac{P_0}{K}\right) + (\alpha_e + \beta_e)\,T\right) / \sqrt{\beta_e T}$, and $\rho(\cdot)$ denotes the standard normal density function.