Stochastic Skew and Target Volatility Options

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Abstract

Target volatility options (TVO) are a new class of derivatives whose payoff depends on some measure of volatility. These options allow investors to take a joint exposure to the evolution of the underlying asset, as well as to its realized volatility. In equity options markets the slope of the skew is largely independent of the volatility level. A single-factor Heston based volatility model can generate steep skew or flat skew at a given volatility level but cannot generate both for a given parameterization. Since the payoff corresponding to TVO is a function of the joint evolution of the underlying asset and its realized variance, the consideration of stochastic skew is a relevant question for the valuation of TVO. In this sense, this article studies the effect of considering a multifactor stochastic volatility specification in the valuation of the TVO as well as forward-start TVO.

Keywords: target volatility options, stochastic volatility, multifactor, stochastic skew, forward-start options.

JEL: G1, G2, G12, G13.

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1 Introduction

In recent years volatility derivatives have become quite popular and liquid. In particular, in 2004 the Chicago Board Options Exchange (CBOE) introduced futures traded on the CBOE Volatility Index (VIX index) and in 2006 options on that index. Interestingly, VIX options are among the most actively trading contracts at CBOE. Target volatility options (TVO) are a new class of derivatives whose payoff depends on some measure of volatility. These options allow investors to take a joint exposure to the evolution of the underlying asset, as well as to its realized volatility. For instance, a target volatility call (TVC) can be viewed as a European call whose notional amount depends on the ratio of the target volatility (a fixed quantity representing the investor's expectation of the future realized volatility) and the realized volatility of the underlying asset over the life of the option. Therefore, when the realized volatility is high, the exposure to the evolution of the asset price is reduced. On the other hand, when the realized volatility is low, the nominal amount invested in the call increases. Since, usually, there is an inverse relationship between the evolution of asset prices and the realized volatility in equity markets, this kind of options allow investors to obtain a greater exposure to the evolution of the underlying asset in a bull market environment.

Di Graziano and Torricelli (2012) provide prices of TVO in the context of single-factor stochastic volatility models when the instantaneous volatility of the underlying asset is assumed to be independent from the Brownian motion driving the asset returns. Unfortunately, this assumption is quite restrictive and it is not realistic. Torricelli (2013) leaves the assumption of independence and allows for correlation between the asset returns and its instantaneous volatility. This author prices TVO under several single-factor stochastic volatility specifications such as the Heston (1993) model or the 3/2 stochastic volatility model. On the other hand, Wang and Wang (2013) study the variance hedging strategy corresponding to TVO.

In equity options markets, the slope of the skew is largely independent of the volatility level. In this sense, there are low volatility days with a steep volatility slope, as well as a flat volatility slope. On the other hand, we also have high volatility days with steep and flat volatility slopes (Derman, 1999). A single-factor Heston based stochastic volatility model can generate steep skew or flat skew at a given volatility level but cannot generate both for a given parameterization. Christoffersen et al. (2009) extend the original Heston (1993) specification to consider a two-factor stochastic volatility model built upon the square root process. This model accounts for stochastic correlation between the asset return and its instantaneous variance. Da Fonseca et al. (2008) consider a Wishart specification to introduce a correlation structure between the single asset noise and the volatility factors. Hence, both models are able to generate stochastic skew. Da Fonseca et al. (2014) extend to the Wishart-based stochastic volatility models the pricing of different volatility-equity options.

Importantly, multifactor stochastic volatility models provide more flexibility not only to model the evolution of the skew/smile but also to model the volatility term structure. In this sense, Marabel Romo (2013a) shows that the consideration of a multifactor stochastic volatility specification has relevant pricing consequences in the valuation of forward skew dependent derivatives such as cliquet and Reverse Cliquet options.

Since the payoff corresponding to TVO is a function of the joint evolution of the underlying asset and its realized variance, the consideration of stochastic skew may have a big impact on the valuation of TVO. In this sense, this article contributes to the literature by studying the effect of considering a multifactor stochastic volatility specification in the valuation of the TVO. To this end, we consider the two-factor Heston-based model of Christoffersen et al. (2009) (TF Heston) in order to investigate TVO. We also provide pricing formulas, under the TF Heston model, corresponding to forward-start TVO, that is, TVO where the strike is determined at a later date. Note that within the single-factor Heston (1993) model, where the correlation between assets returns and the instantaneous variance is constant, the more negative this correlation is, the higher the leverage of the TVO in a bull market and, hence, the higher the price of the TVO, especially for in-the-money TVC. However, under the TF Heston model, where this correlation is stochastic, we will have paths where the asset goes up and the realized volatility goes down with a different instantaneous correlation with respect to the situation of constant correlation. In this case, we should expect pricing discrepancies between these two models.

The rest of the study proceeds as follows. Section 2 studies the sensitivities of TVO with respect to market parameters and develops the pricing formula associated with standard-start, as well as forward-start TVO under the TF Heston model. Section 3 displays the results associated with the pricing tests. Finally, Section 4 offers concluding remarks.

2 Target Volatility Options under Multifactor Stochastic Volatility

2.1 The Target Volatility Option

As said previously, a TVC can be viewed as a European call whose notional amount depends on the ratio of the target volatility and the realized volatility of the underlying asset over the life of the option. Hence, when the realized volatility is high, the exposure to the evolution of the asset price is reduced. Conversely, when the realized volatility is low, the option leverage increases. Since, usually, there is an inverse relationship between the evolution of asset prices and the realized volatility in equity markets, this kind of options allow investors to obtain a greater exposure to the evolution of the underlying asset when it exhibits a positive trend.

Let S_t denote the spot price of the underlying asset at time $t \geq 0$, V_t its instantaneous variance and $Y_t = \ln S_t$ the log-price process. The payoff of a TVC with maturity T, strike price K and target volatility level γ can be expressed as follows:

$$w\left(Y_T, I_T\right) \quad : \quad = \left(S_T - K\right)^+ \gamma \frac{\sqrt{T}}{\sqrt{I_T}} = \left(e^{Y_T} - e^{\ln K}\right)^+ \gamma \frac{\sqrt{T}}{\sqrt{I_T}}, \tag{1}$$

where $I_t = \int_0^t V_s ds$ denotes the integrated variance process. Note that if the realized volatility coincides with the target volatility level, the pay-off of TVC reduces to the payoff of a standard European call. For example, let us assume, for a moment, that we are within the Black-Scholes (1973) (BS) framework. In this case, the payoff of the TVC in equation (1) simplifies to

$$w^{BS}\left(Y_T\right) = \left(e^{Y_T} - e^{\ln K}\right)^+ \frac{\gamma}{\sum^{BS}}$$

with $\sum^{BS}$ being the corresponding BS implied volatility. In the particular case, where the implied volatility $\sum^{BS}$ coincides with the target volatility level, the payoff of a TVC is exactly the same as the payoff associated with a plain vanilla call with the same strike price. Since the payoff of the TVO is given by a joint function of the asset returns and its realized variance, it is necessary to consider the existence of stochastic volatility to price this kind of derivatives.

The Vega of an option represents its sensitivity with respect to the implied volatility. Let $P_{0,TVC}^{BS}$ denote the time $t = 0$ price of the TVC under the BS specification. In this sense, the

Vega of the TVC is given by:

$$\frac{\partial P_{0,TVC}^{BS}\left(K,T\right)}{\partial \Sigma^{BS}} := Vega_{TVC}^{BS} = \frac{\partial P_{0,C}^{BS}\left(K,T\right)}{\partial \Sigma^{BS}}\frac{\gamma}{\Sigma^{BS}} - P_{0,C}^{BS}\left(K,T\right)\frac{\gamma}{\left(\Sigma^{BS}\right)^{2}}$$

where $P_{0,C}^{BS}\left(K,T\right)$ is the BS price of a European call with strike price K and maturity T. Figure 1 displays the Vega of at-the-money TVC, under the BS framework, for different maturities as a function of the implied volatility. The target volatility level γ is equal to 0.25 and the Vega is expressed in percentage terms. We can see from the figure, that the higher the implied volatility the lower the Vega of the TVC. In this sense, let us consider a financial institution that sells a TVC to an investor and let us assume that this institution hedges initially the Vega exposure of this options. If afterwards the implied volatility goes up, the financial institution will be long Vega given that the sensitivity of the Vega with respect to the implied volatility is negative for the TVC. This sensitivity is called Volga or Vomma (see, for example, Castagna and Mercurio, 2007 or Wilmott, 2006). Interestingly, other exotic derivatives, which are usually sold by financial institutions to their clients, such as reverse cliquet options, exhibit Volga sensitivities with opposite sign. Hence, TVO could represent a natural way of hedging these kind of risks which are usually quite hard to hedge in the inter-bank and inter-dealer broker markets. Moreover, this product is pretty interesting for final investors as well, who want to be leveraged in the market upside given that, usually, there is a negative relationship between assets returns and volatility.

As said previously, since the payoff corresponding to TVO is a function of the joint evolution of the underlying asset and its realized variance, the consideration of stochastic skew may have a relevant effect on the valuation of TVO. Hence, we now analyze the pricing of TVO under the TF Heston specification.

2.2 The two-factor Heston specification

Using a principal component analysis of the Black-Scholes (1973) implied variances, Christoffersen et al. (2009) show that the first two components together explain more than 95% of the variation in the data. These results seem to suggest that a two-factor model may be a good choice. We now briefly present the main features of the TF Heston specification. For simplicity, we assume that the continuously compounded risk-free rate r and dividend yield q are constant. Let Q denote the risk neutral probability measure defined on a filtered probability space $(\Omega, \mathcal{F}, Q)$, such that under Q asset prices multiplied by the exponential of cumulated dividends, expressed in terms of the money market account, are martingales. The TF Heston model assumes the following dynamics for the return process Y_t under Q:

$$dY_t = \left[r - q - \frac{1}{2}\sum_{i=1}^{2} v_{it}\right] dt + \sum_{i=1}^{2} \sqrt{v_{it}} dZ_{it}, \tag{2}$$

where the instantaneous variance processes follow the square root dynamics

$$dv_{it} = \kappa_i \left(\theta_i - v_{it}\right) dt + \sigma_i \sqrt{v_{it}} dW_{it}$$

where θ_i represents the long-term mean corresponding to the instantaneous variance factor i (for $i = 1, 2$), κ_i denotes the speed of mean reversion and, finally, σ_i represents the volatility of the variance factor i. For analytical convenience, let us rewrite the previous equation as follows:

$$dv_{it} = \left(a_i - b_i v_{it}\right) dt + \sigma_i \sqrt{v_{it}} dW_{it} \tag{3}$$

where $b_i = \kappa_i$ and $a_i = \kappa_i \theta_i$. In equations (2) and (3) Z_{it} and W_{it} are Wiener processes such that

$$dZ_{it} dW_{jt} = \begin{cases} \rho_i dt \text{ for } i = j, \\ 0 \text{ for } i \neq j. \end{cases}$$

In order to preserve affinity, Z_{1t} and Z_{2t} are uncorrelated. In addition, W_{1t} and W_{2t} are also uncorrelated. The conditional variance of the return process is:

$$V_t := \frac{1}{dt} Var(dY_t) = \sum_{i=1}^{2} v_{it}$$

whereas, as shown by Christoffersen et al. (2009), the correlation between the asset return and the variance process is:

$$\rho_{S_t V_t} := Corr\left(dY_t, dV_t\right) = \frac{\rho_1 \sigma_1 v_{1t} + \rho_2 \sigma_2 v_{2t}}{\sqrt{\sigma_1^2 v_{1t} + \sigma_2^2 v_{2t}} \sqrt{v_{1t} + v_{2t}}}.$$

Importantly, the TF Heston model, unlike the single-factor Heston (1993) model, allows for stochastic correlation between the asset return and the variance process and, hence, it accounts for the existence of stochastic skew consistent with the empirical evidence (see for example Carr and Wu 2007). Note that if we drop the presence of the second factor in the previous expression, the correlation between the asset return and the variance process simplifies to the constant correlation level ρ_1 associated with the Heston (1993) model. In addition, the TF Heston model provides more flexibility to model the volatility term structure than single-factor specifications.

2.3 Pricing target volatility options

In this section we follow the methodology of Lewis (2000) and Da Fonseca et al. (2007) to calculate option prices efficiently in terms of the generalized Fourier transform associated with the payoff function and with the asset return. In this sense, let us consider the payoff corresponding to the TVC $w(Y_T, I_T)$. From the Fundamental Theorem of Asset Pricing, the time $t = 0$ price of this option, denoted TVC_0, is given by:

$$TVC_0(K,T) = e^{-rT}\mathbb{E}^Q\left[w(Y_T, I_T)\right] = e^{-rT}\int_{\mathbb{R}^2} w(Y_T, I_T)\,\delta_T(Y_T, I_T)\,dU_T \qquad (4)$$

with $U_T = (Y_T, I_T)'$ and where $\delta_T(Y_T, I_T)$ is the risk-neutral density function of Y_T and I_T. The joint (extended) Laplace transform of the asset returns and the integrated variance process is defined as:

$$\psi(\lambda_x, \lambda_v; Y_0, I_T, T) := \mathbb{E}^Q\left[e^{\lambda_x Y_T + \lambda_v I_T}\right] = \int_{\mathbb{R}^2} e^{\lambda_x Y_T + \lambda_v I_T}\delta_T(Y_T, I_T)\,dU_T \qquad \lambda_x,\ \lambda_v \in \mathbb{C}.$$

On the other hand, the Fourier transform corresponding to the payoff $w(Y_T, I_T)$ is given by

$$\widehat{w}(z) = \int_{\mathbb{R}^2} e^{iz_1 Y_T + iz_2 I_T} w(Y_T, I_T)\,dU_T \qquad z = (z_1, z_2)' \in \mathbb{C}^2, \qquad (5)$$

with

$$w(Y_T, I_T) = \frac{1}{(2\pi)^2}\int_{\chi} e^{-i\langle z, U_T\rangle}\widehat{w}(z)\,dz,$$

where $\chi \subset \mathbb{C}^2$ is the admissible integration domain in the complex plane corresponding to the generalized Fourier transform associated with the payoff function, $i^2 = -1$ and where $\langle .,. \rangle$ represents the scalar product in $\mathbb{C}^2$. Substituting the previous expression in equation (4) yields:

$$TVC_0(K,T) = \frac{e^{-rT}}{(2\pi)^2}\int_{\chi} \psi(\lambda_x = -iz_1, \lambda_v = -iz_2; Y_0, I_T, T)\,\widehat{w}(z)\,dz, \qquad (6)$$

where we have used the Fubini theorem. Appendix A shows that the Fourier transform associated with the payoff function $w(Y_T, I_T)$ is given by[1]

$$\widehat{w}(z) = \frac{e^{\ln K(1+iz_1)}\sqrt{T\pi}}{iz_1(1+iz_1)\sqrt{2z_2}}(1+i)\gamma \qquad (7)$$

[1]A target volatility put will have the same payoff transform as a call, but a different admissible domain given by $Im(z_1) < 0$.

and

$$\chi = \left\{ z \in \mathbb{C}^2 : \; Im\left(z_1\right) > 1, \; Im\left(z_2\right) > 0 \right\}.$$

Hence, to obtain a semi-closed-form solution for the option price we need to calculate the joint Laplace transform of the asset returns and the integrated variance process. Using standard arguments (Da Fonseca and Grasselli, 2011 or Marabel Romo, 2013a) it is easy to show that, under the risk-neutral measure Q, the Laplace transform associated with the TF Heston model $\psi\left(\lambda_x, \lambda_v; Y_0, I_T, T\right)$ is given by:

$$\psi\left(\lambda_x, \lambda_v; Y_0, I_T, T\right) := \psi\left(\lambda; Y_0, I_T, T\right) = e^{B(\lambda,T) + \sum_{i=1}^{2} A_i(\lambda,T) v_{i0} + \lambda_x Y_0} \qquad \lambda = \left(\lambda_x, \lambda_v\right)', \qquad (8)$$

where the $A_i\left(\lambda, T\right)$, $i = 1, 2$, are given by

$$A_i\left(\lambda, T\right) = \frac{b_i - \lambda_x \rho_i \sigma_i + \alpha_i}{\sigma_i^2} \left[\frac{1 - e^{T\alpha_i}}{1 - \beta_i e^{T\alpha_i}} \right]$$

with

$$\begin{aligned} \alpha_i &= \left[\left(\lambda_x \rho_i \sigma_i - b_i\right)^2 - 2 d_1\left(\lambda\right) \sigma_i^2 \right]^{1/2}; \\ \beta_i &= \frac{b_i - \lambda_x \rho_i \sigma_i + \alpha_i}{b_i - \lambda_x \rho_i \sigma_i - \alpha_i}; \\ d_1\left(\lambda\right) &= \frac{\lambda_x}{2}\left(\lambda_x - 1\right) + \lambda_v. \end{aligned}$$

On the other hand, $B\left(\lambda, T\right)$ can be expressed as:

$$\begin{aligned} B\left(\lambda, T\right) &= d_0\left(\lambda\right) T + \sum_{i=1}^{2} \frac{a_i}{\sigma_i^2} \left[\left(b_i - \lambda_x \rho_i \sigma_i + \alpha_i\right) T - 2 \ln\left(\frac{1 - \beta_i e^{T\alpha_i}}{1 - \beta_i} \right) \right], \\ d_0\left(\lambda\right) &= \lambda_x \left(r - q\right). \end{aligned}$$

Hence if we want to price a standard-start TVC, we can combine equations (6), (7) and (8) to obtain a semi-closed form solution for the price of target volatility calls. From the put-call parity it is easy to obtain the price associated with a European put. Of course, the price of TVO under the classic Heston (1993) model can be recovered by the previous formulas as a particular case.

2.3.1 Pricing forward-start TVO options

In this section we consider the pricing problem associated with TVO when the strike will be determined at a later date. In the context of single assets whose volatility follows a Wishart process, Da Fonseca et al. (2008) provide an explicit expression for the price of forward-start European options. This section considers the pricing problem associated with forward-start TVO options under the TF Heston model. To this end, let us define the forward log-return process as $Y_{t,T} := Y_T - Y_t$ and $I_{t,T} := I_T - I_t$ where

$$I_t = \int_0^t \sum_{i=1}^{2} v_{is} ds$$

denotes the integrated variance process. We consider a forward-start TVC between the strike date t and the maturity date T with payoff:

$$w\left(Y_{t,T}, I_{t,T}\right) := \left(\frac{S_T}{S_t} - K\right)^+ \gamma \frac{\sqrt{T-t}}{\sqrt{I_{t,T}}} = \left(e^{Y_T - Y_t} - e^{\ln K}\right)^+ \gamma \frac{\sqrt{T-t}}{\sqrt{I_{t,T}}},$$

Let us denote by $\psi_{v1,v2}\left(\lambda_x, \lambda_v; t, T\right)$ the forward Laplace transform

$$\psi_{v1,v2}\left(\lambda_x, \lambda_v; t, T\right) := \mathbb{E}^Q\left[e^{\lambda_x (Y_T - Y_t) + \lambda_v (I_T - I_t)}\right] = \mathbb{E}^Q\left[e^{\lambda_x (Y_T - Y_t) + \lambda_v I_{t,T}}\right],$$

which can be expressed as follows:

$$\psi_{v1,v2}\left(\lambda_x, \lambda_v; t, T\right) = \mathbb{E}^Q\left[e^{-\lambda_x Y_t} \mathbb{E}_t^Q\left[e^{\lambda_x Y_T + \lambda_v I_{t,T}}\right]\right],$$

where $\mathbb{E}_t^Q[.]$ denotes the expectation conditional on the information available through time t. We can rewrite the previous expression as:

$$\psi_{v1,v2}\left(\lambda_x, \lambda_v; t, T\right) = \mathbb{E}^Q\left[e^{-\lambda_x Y_t} \psi\left(\lambda; Y_t, I_{t,T}, T-t\right)\right],$$

where

$$\begin{aligned} \psi\left(\lambda; Y_t, I_{t,T}, T-t\right) : &= \mathbb{E}_t^Q\left[e^{\lambda_x Y_T + \lambda_v I_{t,T}}\right] \\ &= e^{B(\lambda, T-t) + \sum_{i=1}^{2} A_i(\lambda, T-t) v_{it} + \lambda_x Y_t} \end{aligned}$$

is the usual joint Laplace transform associated with standard-start options. Hence,

$$\psi_{v1,v2}\left(\lambda_x, \lambda_v; t, T\right) = e^{B(\lambda, T-t)} \mathbb{E}^Q\left[e^{\sum_{i=1}^{2} A_i(\lambda, T-t) v_{it}}\right].$$

Let $\Psi(H_1, H_2; V_0, t) := \Psi(t)$ denote the Laplace transform corresponding to the variance process V_t, under Q defined as:

$$\Psi(t) := \mathbb{E}^Q\left[e^{\sum_{i=1}^2 H_i v_{it}}\right] \qquad H_1, H_2 \in \mathbb{C}.$$

If we set $H_i = A_i(\lambda, T-t)$, using standard arguments (see e.g. Da Fonseca and Grasselli 2011 or Marabel Romo 2013a) it is easy to show that

$$\Psi(A_1(\lambda, T-t), A_2(\lambda, T-t); V_0, t) = e^{M(t)+\sum_{i=1}^2 N_i(t) v_{i0}},$$

where

$$N_i(t) = \frac{A_i(\lambda, T-t)\, e^{-tb_i}}{1 + \frac{\sigma_i^2}{2}\frac{A_i(\lambda,T-t)}{b_i}\left[e^{-tb_i} - 1\right]} \qquad i = 1, 2,$$

and where $M(t)$ can be obtained as follows:

$$M(t) = -2\sum_{i=1}^{2}\frac{a_i}{\sigma_i^2}\ln\left[1 + \frac{\sigma_i^2}{2}\frac{A_i(\lambda, T-t)}{b_i}\left(e^{-tb_i} - 1\right)\right].$$

In conclusion, the forward Laplace transform required to price forward-start TVO is given by

$$\psi_{v1,v2}(\lambda_x, \lambda_v; t, T) = e^{B(\lambda, T-t)+M(t)+\sum_{i=1}^2 N_i(t) v_{i0}}, \tag{9}$$

so that the initial price corresponding to the forward-start TVC is given by:

$$FSTVC_0(t, T, K) = \frac{e^{-rT}}{(2\pi)^2}\int_{\chi}\psi_{v1,v2}(\lambda_x = -iz_1, \lambda_v = -iz_2; t, T)\,\widehat{w}(z)\,dz. \tag{10}$$

In the context of European options under the Wishart specification, Da Fonseca et al. (2008) show that the value of forward-start options is independent on the level of the underlying asset and depends only on the volatility process. In the case of forward-start TVO options under the TF Heston model, from equation (9), the forward Laplace transform does not depend on the asset price and, hence, the price of forward-start TVO depends only on the volatility process as well: this is a peculiarity of affine specifications of the model. For example, in the non-affine framework where the underlying asset displays a mean reversion, Marabel Romo (2013b) shows the value of forward-start European options is also affected by the underlying asset price.

3 Single-Factor Versus Multifactor Stochastic Volatility Models in the Pricing of TVO

3.1 Market data calibration

When we use a particular model to price exotic options without liquid markets, such as TVO, practitioners calibrate the model parameters to the market price of plain vanilla options at a point in time and use the model parameters to price exotic options at the same time. The market practice is to obtain a calibration per day. In this sense, to compare the pricing performance of single-factor and multifactor stochastic volatility models in the valuation of TVO, we calibrate the parameters of the Heston (1993) model and the TF Heston model to the market prices of plain vanilla options corresponding to February 8, 2012 and November 7, 2013 for Google Inc., obtained from Bloomberg. This company is ranked third in terms of weighted capitalization within the Dow Jones Global Titans index which incorporates the 50 largest multinational companies. Google Inc. does not pay dividends and, hence, conversely to those companies that pay discrete dividends, it is easy to incorporate this dividend policy within our framework just setting the dividend yield q equal to zero.

The data consist of prices associated with listed options corresponding to eleven maturities. The maturities range from April 2012 to February 2017 for February 8, 2012 and from January 2014 to November 2018 for November 7, 2013. For each maturity, we consider thirteen values of moneyness[2], ranging from 70% to 130%. Hence, a total of 143 points on the implied volatility surface are provided for each date.

Figure 2 shows the data graphically for each of the dates considered. In both cases, the implied volatility surface exhibits a negative volatility skew, which is most pronounced for near-term options. This is a common pattern of behavior that has been widely observed in the equity options market. Some evidence can be found, among others, in Derman et al. (1995) and Gatheral (2006). Importantly, although the skew is omnipresent in the equity options market, the implied volatility surface evolves stochastically through time, thus generating Vega risk. In this sense, the at-the-money implied volatility associated with November 7, 2013 is 2.70% lower in average than the at-the-money implied volatility corresponding to February 8, 2012. We consider two dates with different volatility regimes to investigate if there are significant pricing discrepancies between the TF Heston and the single-factor Heston (1993) specifications in the

[2] The moneyness is defined as $\frac{K}{S}$ where K is the strike price and S is the current spot price.

pricing of TVO, as well as to investigate if the side and magnitude of these discrepancies are robust to different market conditions.

To improve the calibration of short term implied volatility the market practice consists in putting more weight on short maturity options using the inverse of the Vega. Along the lines of Christoffersen et al. (2009) and Da Fonseca and Grasselli (2011), among others, in this study we follow this approach and we choose the model parameters that solve the following optimization problem:

$$\min_{\xi} \frac{1}{N_i N_j} \sum_{i=1}^{N_i} \sum_{j=1}^{N_j} \left[\frac{C^{mkt}\left(K_i, T_j\right) - C_\gamma\left(K_i, T_j\right)}{\varsigma\left(K_i, T_j\right)} \right]^2,$$

where ξ is the vector of parameters to be estimated, $C^{mkt}\left(K_i, T_j\right)$ is the market price of a European call with strike price K_i and maturity T_j, $C_\gamma\left(K_i, T_j\right)$ is the model price, $\varsigma\left(K_i, T_j\right)$ is the Vega corresponding to a European option with strike K_i and maturity T_j, N_i is the total number of strikes and, finally, N_j represents the number of maturities considered.

Table 1 displays the calibrated parameters values and the mean absolute errors (MAE) associated with implied volatilities corresponding to February 8, 2012 under both models, while table2 provides the estimation results corresponding to November 7, 2013. In both cases, the MAE associated with the Heston (1993) model, as well as with the TF Heston model, are of the same order of magnitude as the calibration errors obtained by Da Fonseca and Grasselli (2011) for some equity indices.

Panel A of figure 3 provides the implied volatility surfaces generated by both models corresponding to February 8, 2012, whereas panel B shows the implied volatility differences. Figure 4 displays the same data for November 7, 2013. A negative value means that the implied volatility is lower under the TF Heston specification than under the single-factor specification. We can see from the figures that, although the implied volatility differences are generally small, the main discrepancies correspond to out-of-the-money puts with short-term maturities where the implied volatility associated with the TF Heston model is lower than under the single-factor specification.

Note that the Feller's condition[3], which ensures that the variance factors do not reach zero, is not satisfied for the TF Heston model associated with November 7, 2013. Christoffersen et al. (2009) and Da Fonseca and Grasselli (2011), among others, obtained similar results in the calibration of the Heston (1993) model and other multifactor stochastic volatility models

[3]The Fellers condition is given by $2\kappa_i\theta_i \geq \sigma_i^2$.

using equity options data. As pointed out by Da Fonseca and Grasselli (2011), one possible explanation is that the mean reverting parameter is problematic to estimate and uses to be small. This problem is mainly related to the fact that option prices contain integrated volatility. One consequence is that the volatility remains too close to zero and contrasts with its empirical distribution which is closer to a log-normal one. In this sense, we are not surprised by this property though it might be a problem in some practical situations. For instance, there are derivatives whose payoff depends on the time spent by the volatility process close to zero. Those products will be mispriced by the above models and might justify the market practice to use processes for the volatility that are positive by construction. But,in our case, the idea is to calibrate the parameters to be as consistent as possible with the market prices of vanilla options. If we include restrictions within the calibration process the results may be not desirable in terms of the ability of the model to replicate the prices of observed plain vanilla instruments.

3.2 Pricing tests

In this section, we study the pricing results of the single-factor Heston (1993) model when compared with the TF Heston specification in the valuation of the TVC. In this sense, panel A of table 3 displays the prices, calculated using equation (6), corresponding to TVC with maturity within six months for different strike prices associated with market data corresponding to February 8, 2012. On the other hand, panel B of table 3 provides the prices associated with options with maturity within one year. Analogously, table 4 provides the same information for November 7, 2013.

The integral in equation (6) is evaluated using a multidimensional global adaptive strategy, truncating the integral limits at 100. Increasing the truncation of the integral does not change the pricing results. We have performed several numerical simulation tests to verify that the option prices obtained from the semi-closed-form solution presented in this article match with those produced with Monte Carlo simulations. Regarding the Monte Carlo specification, we consider daily time steps and 50,000 trials and we implement the quadratic exponential scheme as described in Andersen (2008).

We can see from tables 3 and 4 that the prices associated with at-the-money and out-of-the-money options are similar under both models. However, the prices corresponding to in-the-money TVC are lower under the TF Heston model than under the single-factor specification for both maturities and dates, especially for short-term maturities. Within the single-factor Heston

(1993) specification, where the correlation between assets returns and the instantaneous variance is constant, the sensitivity of the price corresponding to TVO with respect to this parameter is not constant across moneyness. In this sense, for in-the-money TVC the more negative the correlation between assets returns and the instantaneous variance, the higher the likelihood that the call remains in-the-money and the higher the leverage of the TVO in a bull market. Hence, in this case, the price of the TVC will be higher. Regarding out-of-the-money TVC the following question arises: what is the effect of this correlation coefficient for out-of-the-money TVC? Is it the same as the one associated with in-the-money TVC? Intuitively, we have two different effects here. On the one hand, the more negative the correlation between assets returns and the variance process for out-of-the-money TVC the lower the likelihood that the option becomes in-the-money since the implied volatility associated with out-of-the-money plain vanilla calls will be really low. This effect tends reduce the price of the TVC. On the other hand, the more negative the correlation between assets returns and the instantaneous variance the higher the leverage of the TVC. This effect tends to increase the price of the TVC. But if the likelihood that the option becomes in-the-money is low, increasing the leverage will not be quite relevant. Under the assumption that the first effect dominates, we will have a situation where reducing the correlation between assets returns and the variance process increases the price associated with int-the-money TVC while, at the same time, reduces the price corresponding to out-of-the-money TVC. To analyze this argument, we consider the estimated parameters associated with the Heston model for November 7, 2013 and we price TVC with maturity within one year when we use $\rho_1 = -0.81$ instead of the initial estimated parameter. Table 7 displays the pricing results. The comparison of tables 4 and 7 shows that, for this parameter combination, the price corresponding to in-the-money TVC is higher, whereas the price associated with out-of-the-money TVC is lower. Under stochastic correlation, following the previous argument, if there are paths where the asset goes up and the realized volatility goes down with a greater correlation (less negative) than in the case of constant correlation, we should expect that the in-the-money TVC will be less expensive under the TF Heston framework whereas, at the same time, we could find cases where out-of-the-money TVC are more expensive under the TF Heston model than under the single-factor Heston (1993) specification.

Panel A of table 5 shows the prices associated with at-the-money forward-start TVO under both models corresponding to February 8, 2012, whereas panel B displays the pricing results associated with November 7, 2013. We assume that the maturity is equal to six months and we calculate the price of forward-start TVC starting within three months, as well as within six months. To this end, we use the pricing formula of equation (10). We can see from the table that the price differences corresponding to the TF Heston model and to the single-factor Heston (1993) specification are similar to the price discrepancies that we obtained for standard-start TVC with maturity within six months. In this sense, the model discrepancies associated with standard-start TVO are also preserved for forward-start options.

The introduction of additional volatility factors in the context of stochastic volatility models allows us to generate more flexible smile patterns that improve the empirical fit of the model to the market price of European options. Note that the payoff corresponding to TVO is a function of the joint evolution of the underlying asset and its realized variance. Importantly, the previous results tend to indicate that the introduction of these additional volatility factors and, hence, the consideration of stochastic skew has relevant effects on the pricing performance of TVO and, in particular, for in-the-money TVC.

3.2.1 Discretely sampled target volatility options

The pricing formula for TVO developed within this study considers continuous sampling of the realized variance processes whereas, generally, contractual specifications compute the realized variance based on sampling at discrete times. In this case the payoff associated with a TVC becomes:

$$w\left(Y_T, I_T\right) := \left(S_T - K\right)^+ \gamma \frac{\sqrt{T}}{RV_T}, \tag{11}$$

where

$$RV_T = \sqrt{\sum_{i=0}^{N-1} \ln\left[\frac{S_{i+1}}{S_i}\right]^2}$$

represents the realized volatility of the underlying asset over the lifetime of the option.

The TVO is basically an hybrid product of a standard equity option and a volatility derivative. It has been shown that volatility derivatives, such as volatility swaps, may be quite sensitive to the monitoring frequency. For instance, Zheng and Kwok (2012) showed that short-maturity variance swaps are very sensitive to the monitoring frequency. In the case of TVO the standard practice is to consider a daily sampling frequency. Since this question may be of great interest for practitioners, this section analyzes if there are relevant price discrepancies between the continuous version corresponding to the pricing formula of equation (6) and the usual market standard. To this end, table 6 provides the prices associated with TVC, with maturity within one year, corresponding to the Heston (1993), as well as to the TF Heston specification obtained using Monte Carlo simulations based on the payoff of equation (11) considering daily sampling frequency for November 7, 2013. As before, we consider 50,000 trials and we implement the quadratic exponential scheme as described in Andersen (2008).

The comparison of the pricing results associated with tables 4 and 6 shows that, in the case of daily frequency, we do not find relevant discrepancies between the continuous time pricing formula and the Monte Carlo specification[4]. Hence the pricing formula of equation (6) can be considered a reliable instrument to price daily sampled TVO.

4 Conclusion

Target volatility options (TVO) are a new class of derivatives whose payoff depends on some measure of volatility. These options allow investors to take a joint exposure to the evolution of the underlying asset, as well as to its realized volatility. In the case of a target volatility call (TVC), when the realized volatility is high, the exposure to the evolution of the asset price is reduced. Conversely, when the realized volatility is low, the nominal amount invested in the call increases. Since, usually, there is an inverse relationship between the evolution of asset prices and the realized volatility in equity markets, this kind of option allow investors to obtain a greater exposure to the evolution of the underlying asset in a bull market environment.

As said in the introduction, empirical evidence tends to suggest that the shape of the skew is quite independent of the level of volatility. In this sense, we can find low volatility days with a steep volatility skew, as well as low volatility days with a flat volatility skew. Analogously, there are also high volatility days with steep volatility skews, as well as high volatility days with flat volatility skews. In the case of the single-factor Heston (1993) model, the correlation between the underlying asset return and the variance factor is constant. This fact limits the ability of this kind of models to characterize the stochastic nature of the volatility skew. Importantly, the payoff corresponding to TVO is a function of the joint evolution of the underlying asset and its realized variance. Hence, the consideration of stochastic skew may have a relevant effect on the valuation these kind of options.

This article addresses the effects of the consideration of stochastic skew on the valuation of standard-start, as well as forward-start TVO. To this end, we consider the Heston (1993)

[4] Although not reported here for the sake of brevity, we obtain similar results for options with maturity within six months, as well as for the prices of TVO corresponding to February 8, 2012.

model and the extended version proposed by Christoffersen et al. (2009), denoted the Two-Factor Heston (TF Heston) model. To compare the pricing performance of single-factor and multifactor stochastic volatility models in the valuation of TVO, we calibrate the parameters of the Heston (1993) model and the TF Heston model to the market prices of plain vanilla options corresponding to February 8, 2012 and November 7, 2013 for Google Inc. The empirical results tend to indicate that the consideration of additional volatility factors has relevant effects on the pricing performance of standard-start, as well as on forward-start TVO, especially for in-the-money TVC. In this sense, similar calibrations of single-factor and multifactor stochastic volatility models to the current market prices of plain vanilla options can lead to important discrepancies in the pricing of TVO. This fact is related to the existence of model risk, which is the risk that arises from the utilization of an inadequate model. In this case, the key question is: is the TVO quoted in liquid markets or is it only traded over-the-counter without published prices available? If the answer is that it is quoted in liquid markets, hence, by definition, there is no model risk. Traders will calibrate their models to the market prices and all the models will give the same price. In this situation, an interesting question would be if the market prices of these options could be consistent with the prices of vanilla options. But the real answer is that TVO are not liquid. Hence, the model risk is clearly present. In this case, a trader that only sees prices of vanilla options and is asked to price a TVO should care about this issue because she/he may misprice the option. The choice of the correct model to price TVO is still an open question. However, empirical evidence shows that the skew is stochastic and that the TF Heston accounts for this fact whereas the single-factor does not.

Target volatility options represent a relatively new product and are traded mainly within the over-the-counter market. This market is relatively liquid in Europe and especially in Asia although it is hard to obtain information regarding total volumes traded. Recently TVO have started trading within the inter-dealer broker markets. Hopefully the liquidity of this kind of options will increase in the inter-dealer broker markets. This increase of liquidity would help to reduce the model risk associated with the valuation of this kind of structures.

A The Fourier Transform of the Target Volatility Call Payoff

From equation (5) the transform $\widehat{w}(z)$ can be expressed as:

$$\begin{aligned}
\widehat{w}(z) &= \gamma\sqrt{T}\int_0^\infty\int_0^\infty e^{iz_1Y_T+iz_2I_T}\frac{\left(e^{Y_T}-e^{\ln K}\right)^+}{\sqrt{I_T}}dY_TdI_T \\
&= \gamma\sqrt{T}\int_0^\infty e^{iz_1Y_T}\left(e^{Y_T}-e^{\ln K}\right)^+dY_T\int_0^\infty\frac{e^{iz_2I_T}}{\sqrt{I_T}}dI_T \\
&= \gamma\sqrt{T}F_1F_2,
\end{aligned}$$

with

$$\begin{aligned}
F_1 &= \int_0^\infty e^{iz_1Y_T}\left(e^{Y_T}-e^{\ln K}\right)\mathbf{1}_{(Y_T>\ln K)}dY_T; \\
F_2 &= \int_0^\infty\frac{e^{iz_2I_T}}{\sqrt{I_T}}dI_T,
\end{aligned}$$

where $\mathbf{1}_{(.)}$ is the indicator function.

Now

$$F_1 = \int_{\ln K}^\infty\left[e^{iz_1Y_T}\left(e^{Y_T}-e^{\ln K}\right)\right]dY_T,$$

and provided that $Im(z_1)>1$ the previous expression is well defined giving

$$F_1=\frac{e^{\ln K(1+iz_1)}}{iz_1(1+iz_1)}.$$

On the other hand, if we assume that $Im(z_2)>0$, then the expression giving F_2 is well defined and we obtain

$$F_2 = \frac{\sqrt{\pi}}{\sqrt{-iz_2}}=\sqrt{\frac{\pi}{2z_2}}(1+i)$$

Hence, we have

$$\widehat{w}(z)=\frac{e^{\ln K(1+iz_1)}\sqrt{T\pi}}{iz_1(1+iz_1)\sqrt{2z_2}}(1+i)\gamma,$$

within the admissible domain defined by

$$\chi=\left\{z\in\mathbb{C}^2:\ Im(z_1)>1,\ Im(z_2)>0\right\}.$$

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Table 1: Estimation results for the TF Heston model and the Heston (1993) model associated with the implied volatility surface of Google Inc. corresponding to February 8, 2012.

Panel A: TF Heston model			
Parameter	Value	Parameter	Value
κ_1	1.5206	κ_2	2.6758
θ_1	0.0606	θ_2	0.0401
σ_1	0.5945	σ_2	0.4505
ρ_1	-0.7030	ρ_2	-0.1504
v_1	0.0187	v_2	0.0335
MAE implied volatilities	0.3323%		
Panel B: Heston model			
Parameter	Value	Parameter	Value
κ_1	2.0969	ρ_1	-0.3906
θ_1	0.0953	v_1	0.0506
σ_1	0.6033		
MAE implied volatilities	0.3739%		

Table 2: Estimation results for the TF Heston model and the Heston (1993) model associated with the implied volatility surface of Google Inc. corresponding to November 7, 2013.

Panel A: TF Heston model			
Parameter	Value	Parameter	Value
κ_1	2.8131	κ_2	2.7429
θ_1	0.0332	θ_2	0.0421
σ_1	0.4820	σ_2	0.5870
ρ_1	-0.6545	ρ_2	-0.3351
v_1	0.0159	v_2	0.0237
MAE implied volatilities	0.3072%		
Panel B: Heston model			
Parameter	Value	Parameter	Value
κ_1	2.4484	ρ_1	-0.4157
θ_1	0.0772	v_1	0.0397
σ_1	0.6080		
MAE implied volatilities	0.3137%		

Table 3: Prices corresponding to standard-start target volatility calls associated with market data corresponding to February 8, 2012.

Panel A: Options with maturity within six months		
Strike price	TF Heston model	Heston model
0.85	19.26	20.01
1	7.89	8.06
1.15	2.09	2.10
Panel B: Options with maturity within one year		
Strike price	TF Heston model	Heston model
0.85	20.90	21.39
1	11.15	11.31
1.15	5.11	5.01

Notes. Option prices are expressed in percentage.

Table 4: Prices corresponding to standard-start target volatility calls associated with market data corresponding to November 7, 2013.

Panel A: Options with maturity within six months		
Strike price	TF Heston model	Heston model
0.85	21.02	21.88
1	8.06	8.14
1.15	1.82	1.79
Panel B: Options with maturity within one year		
Strike price	TF Heston model	Heston model
0.85	22.43	22.74
1	11.35	11.38
1.15	4.70	4.58

Notes. Option prices are expressed in percentage.

Table 5: Prices corresponding to at-the-money forward-start target volatility calls.

Panel A: Market data corresponding to February 8, 2012		
Maturity: six months		
Forward-start (t)	TF Heston model	Heston model
0.25	7.92	8.00
0.50	7.90	7.98
Panel B: Market data corresponding to November 7, 2013		
Maturity: six months		
Forward-start (t)	TF Heston model	Heston model
0.25	8.09	8.11
0.50	8.07	8.10

Notes. Option prices are expressed in percentage.

Table 6: Prices corresponding to, discretely monitored with daily sampling frequency, standard-start target volatility calls associated with market data corresponding to November 7, 2013

Maturity: one year		
Strike price	TF Heston model	Heston model
0.85	22.44	22.81
1	11.39	11.42
1.15	4.63	4.56

Notes. Option prices are expressed in percentage.

Table 7: Prices corresponding to standard-start target volatility calls associated with market data corresponding to November 7, 2013 when we use $\rho_1 = -0.81$

Maturity: one year	
Strike price	Heston model
0.85	25.18
1	12.80
1.15	4.49

Notes. Option prices are expressed in percentage.

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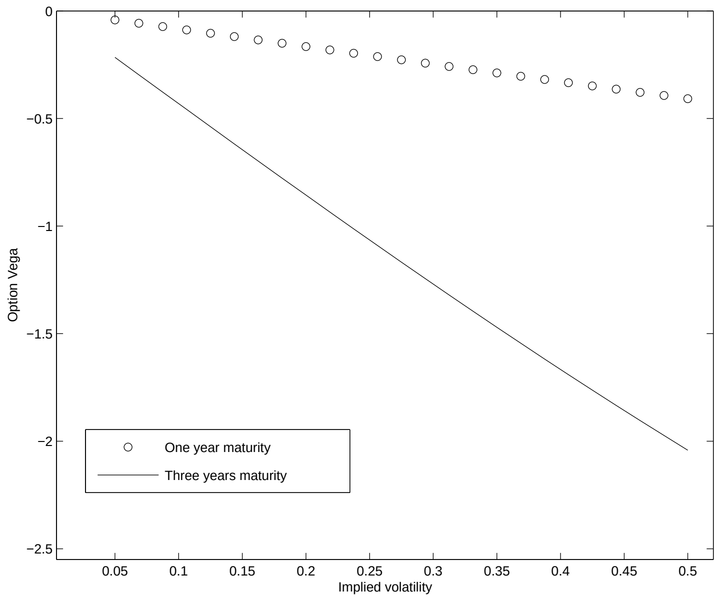

Figure 1: Vega associated with at-the-money standard-start target volatility calls, under the Black-Scholes (1973) framework, for different maturities as a function of the implied volatility. The target volatility level γ is equal to 0.25 and the Vega is expressed in percentage terms.

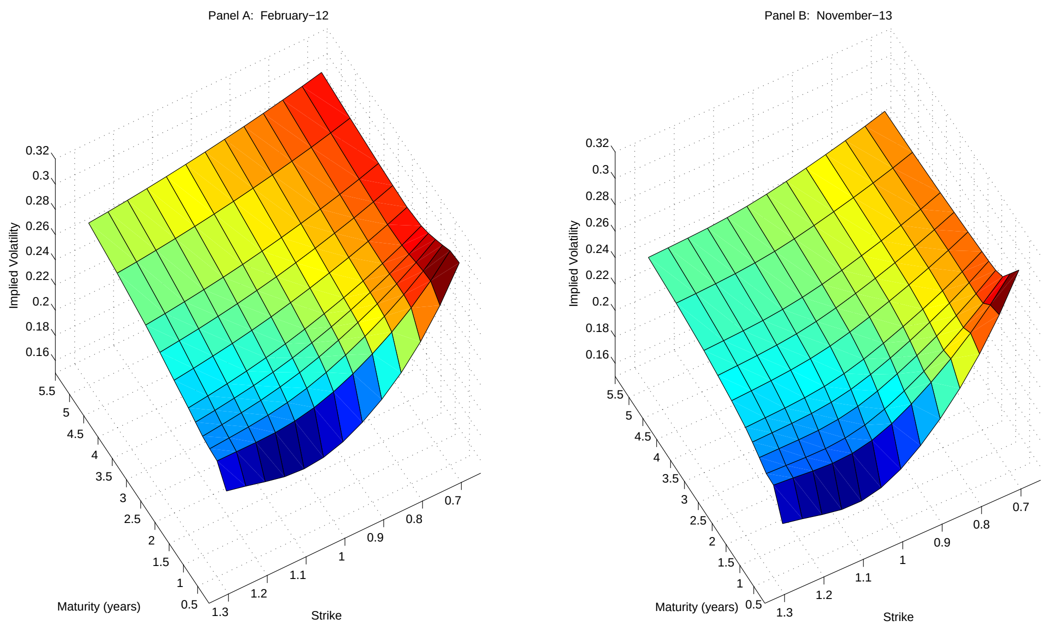

Figure 2: Market implied volatility surfaces for February 8, 2012 (panel A) and for November 7, 2013 (panel B), corresponding to Google Inc. Strike prices are expressed as a percentage of the asset price.

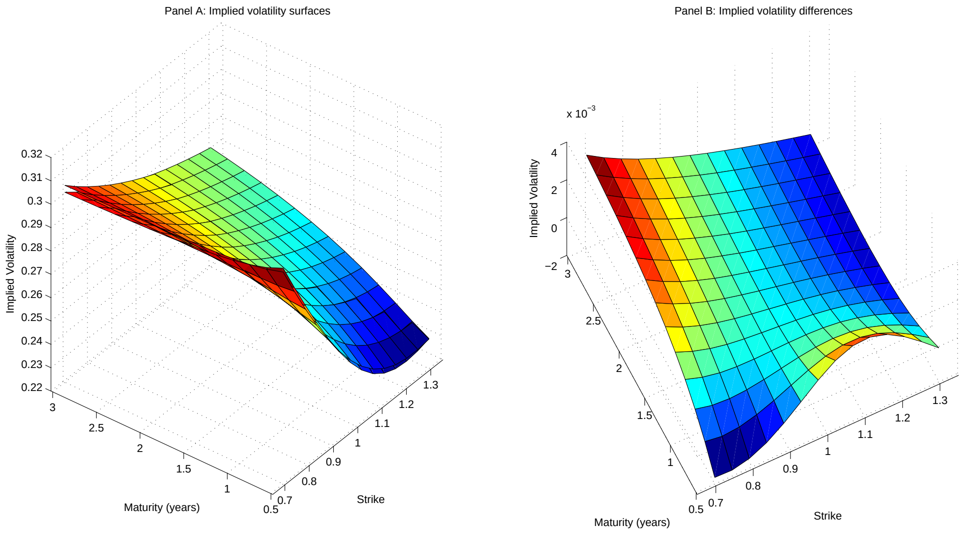

Figure 3: Implied volatility surface generated by the TF Heston specification of table 1, as well as by the Heston model (panel A) and differences between the implied volatility surfaces generated by both specifications (panel B) corresponding to February 8, 2012 for Google Inc. A negative value means that the implied volatility under the TF Heston specification is lower than the implied volatility associated with the standard Heston specification. Strike prices are expressed as a percentage of the asset price, whereas maturities are expressed in years.

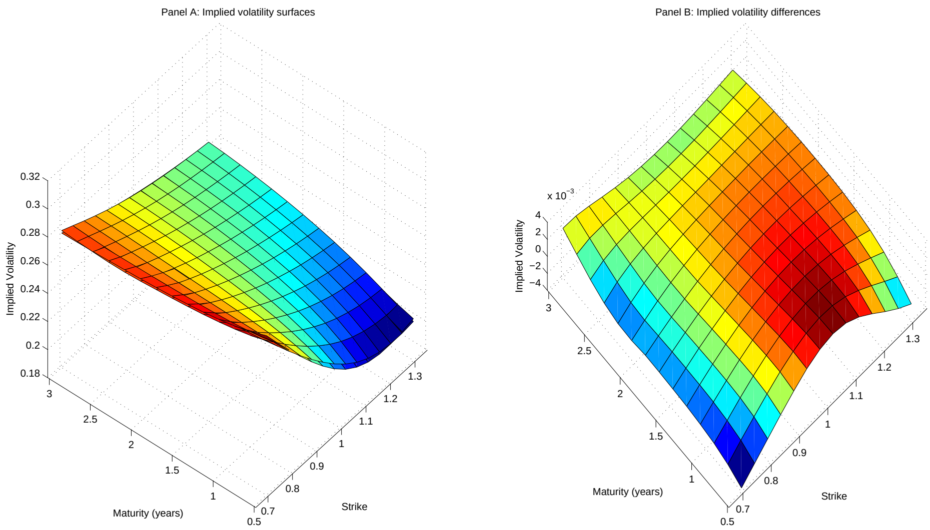

Figure 4: Implied volatility surface generated by the TF Heston specification of table 2, as well as by the Heston model (panel A) and differences between the implied volatility surfaces generated by both specifications (panel B) corresponding to November 7, 2013 for Google Inc. A negative value means that the implied volatility under the TF Heston specification is lower than the implied volatility associated with the standard Heston specification. Strike prices are expressed as a percentage of the asset price, whereas maturities are expressed in years.

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