Bond Market Completeness under Stochastic Strings with Distribution-Valued Strategies

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Abstract

We study bond market completeness under infinite-dimensional models and show that, with stochastic string models, the market is complete if we consider strategies as generalized functions. We also obtain completeness for infinite-dimensional HJM models within the stochastic string framework. This result is not at odds with the incompleteness obtained in Barski *et al.* (2011). We design the hedging portfolio for a wide class of options. We study the Gaussian case and find a closed formula for some compound options. Finally, we prove that the uniqueness of the martingale measure is equivalent to a condition on the set of specific market risk premia.

Keywords: Completeness, stochastic string, generalized function, hedging, martingale measure, bond market, term structure of interest rates.
JEL classification: G11, G13.

1 Introduction

Completeness, i.e., the possibility of replicating any contingent claim with a self-financing portfolio, is a key property in theoretical finance. The equivalence between completeness and uniqueness of the equivalent martingale measure, a result known as Second Fundamental Theorem of Asset Pricing, holds in general securities markets with a finite number of assets (Harrison and Pliska (1981)).

In arbitrage-free models in which the price processes of the k underlying assets are driven by n independent Wiener processes, the model is complete if and only if $k = n$ and the volatility matrix is invertible (Björk (2004), Proposition 14.6). So, informally, we can say that completeness holds whenever the number of risky assets equals the number of sources of randomness. Since continuous-time models of bond markets assume the existence of a continuum of bonds indexed by their maturities, it seems reasonable to think that, to achieve completeness, we need to develop models with an infinite number of sources of randomness. This was done by Björk, Di Masi, Kabanov and Runggaldier (1997) (BDKR), Carmona and Tehranchi (2004), De Donno (2004), De Donno and Pratelli (2004), and Barski *et al.* (2011), among others.

All of these papers share some common features that characterize the state of the art of completeness in infinite-dimensional models. First, all of them must provide a correct definition of infinite-dimensional portfolios and, consequently, an appropriate stochastic integration theory. BDKR consider as integrators (that is, as price processes), stochastic processes that take values in $\mathcal{C}(I)$, with I a compact subset of $[0, +\infty)$, and as integrands (that is, as strategies), processes that take values in the dual of $\mathcal{C}(I)$, i.e., the set of Radon measures on I. In Carmona and Tehranchi (2004) the space for the price processes is an appropriate weighted Sobolev space F, and the strategies are F^*-valued processes belonging to the closure (in the topology of F^*) of the set of finite linear combinations of Dirac deltas.

De Donno and Pratelli (2004) prove that measure-valued strategies are not sufficient to describe all possible portfolios in the market and generalize measure-valued strategies to processes which take values in a Hilbert space called "covariance space". As integrators, they take cylindrical martingales with values in a space of continuous functions. De Donno (2004) uses price processes as semimartingales taking values in $\mathcal{C}([0,1])$ and defines generalized integrands as limits, in the set of operators in $\mathcal{C}([0,1])$, of finite linear combinations of Dirac deltas (the simple integrands). Barski *et al.* (2011) assume that discounted price processes take values in the Sobolev space $G = H^1\left[0, \widehat{T}\right]$, being $\widehat{T}$ the finite time horizon, and consequently, strategies are G^*-valued processes.

One of the important features in the study of completeness with infinite-dimensional models is the fact that if we want our model to explain the "real" portfolios, the strategies proposed have to nest strategies consistent in a finite number of bonds. In the papers just mentioned this is obtained by taking into account that in all cases a strategy can be approximated by finite linear combinations of Dirac measures. In our paper, we will introduce a new approach for modeling infinite-dimensional portfolios. We will not need to use sophisticated techniques of functional analysis. It will suffice to employ a classical topic from this field: Schwartz's Distribution Theory (Schwartz (1966)). By taking strategies as generalized functions, we will use Dirac deltas and we will not need to develop a new stochastic integration theory.

The second common feature of these papers is that none of them demonstrate the completeness of the market. At most, they attain approximate completeness, i.e., they are able to find a sequence of hedgeable contingent claims that converges in some sense to every contingent claim. To the best of our knowledge, this characteristic is shared by all general infinite-dimensional models in the literature (see, for example, Björk, Kabanov and Runggaldier (1997) (BKR) and Taflin (2005)). In contrast, in this paper we obtain market completeness within the model and we are able to show explicitly the replicating strategy.

In some papers (BDKR, BKR, Barski *et al.* (2011)) the completeness of the market is related to the surjectiveness of certain operators, called *hedging operators* in BDKR and BKR. We elaborate on this operational approach below. Other authors obtain conditions under which a contingent claim can be hedged and study the possibility of *natural strategies* in infinite-dimensional models, i.e., strategies composed of bonds with maturities less than or equal to the longest maturity of the bonds underlying the claim (Barski *et al.* (2011)).

Carmona and Tehranchi (2004) pointed out a problem related to multi-factor HJM models: every contingent claim can be hedged by a portfolio of bonds with arbitrary maturities chosen a priori. This is contrary to what traders do in practice, since hedging portfolios are related to the contingent claim being hedged. These authors propose an infinite-dimensional Markovian HJM model and show that, for Lipschitz claims, there exists a hedging strategy consisting of bonds with maturities that are less than or equal to the longest maturity of the bonds underlying the claim. De Donno and Pratelli (2004) also address this problem within the Gaussian framework of Kennedy (1994). They find that a contingent claim on a finite number of bonds can be replicated with a portfolio based on the same bonds and on the bank account.

As an example and without any further assumption, we obtain explicitly a replicating strategy for a very general type of European contingent claims. This strategy consists only of bonds with the same maturities of the bonds underlying the claim.

The third common characteristic in the papers mentioned earlier is that the Second Fundamental Theorem is no longer true for general infinite-dimensional models. The uniqueness of the martingale measure is equivalent only to approximate completeness (BDKR, De Donno and Pratelli (2004)). It is also shown in other papers as BKR and Jarrow and Madam (1999).[1] Taking into account

[1] Jarrow and Madam (1999) prove a version of the Second Fundamental Theorem that states the equivalence between

these results, we study the uniqueness of the equivalent martingale measure regardless of market completeness. We obtain a new result relating uniqueness with the set of possible specific market prices of risk.

To summarize, the main objective of this paper is to state the completeness of the bond market under the stochastic string framework with distribution-valued strategies.

The paper is organized as follows. Section 2 presents the necessary definitions about distributions and portfolios. Section 3 states the main results of completeness for stochastic string models and for HJM models within them. Explicit formulas for the hedges are obtained when it is possible. Section 4 is dedicated to the hedging of European options with degree-one homogeneous payoff functions. In the Gaussian case explicit formulas are obtained and they are applied to the hedging and pricing of call on call options. In Section 5 the uniqueness of the martingale measure in the stochastic string model is studied. It is shown that uniqueness is equivalent to a condition on the form of the possible specific market risk premia. Finally, the main conclusions are presented in Section 6.

2 Definitions

We start by stating some well known concepts related to distribution theory.

Definition 1 *Let $\Omega \subset \mathbb{R}$ be an open subset. The set of* ***test functions*** *over Ω, $\mathcal{D}(\Omega)$, is the linear space of infinitely derivable real functions with compact support included in Ω. The set $\mathcal{D}(\Omega)$ can be equipped with a notion of convergence that makes it a complete locally convex topological vector space. Its topological dual, $\mathcal{D}'(\Omega)$, is called the space of* ***distributions*** *over Ω.* ■

For every function u locally integrable in Ω, the map $T_u : \mathcal{D}(\Omega) \to \mathbb{R}$ defined by $T_u(\varphi) = \int_\Omega u(x)\varphi(x)dx$ is a distribution over Ω. The reciprocal is not true in general. If $T \in \mathcal{D}'(\Omega)$, there is no guarantee that there exists a function $v : \Omega \to \mathbb{R}$ such that the action of T can be written in the form

$$T(\varphi) = \int_\Omega v(x)\varphi(x)dx \tag{1}$$

Nevertheless, by abuse of notation, it is usual in many applications of distribution theory to write every distribution in the form (1) and to identify the distribution T with the so called *generalized function* v. A typical example is the distribution called *Dirac delta*, defined by $T_{\delta_a}(\varphi) = \varphi(a)$, with $a \in \Omega$ that usually is written as $T_{\delta_a}(\varphi) = \int_\Omega \delta(x-a)\varphi(x)\,dx$. We will follow this abuse of notation because it simplifies the calculus and helps us to obtain more financial intuition when working with infinite-dimensional portfolios.

uniqueness of the martingale measure and market completeness just if a certain *fundamental operator* is open. We will return to this result in Section 5.

Now we adapt to our purposes some of the definitions of BKR related to bond portfolios.

Definition 2 *A* ***portfolio*** *in the bond market is a pair* $\{g_t, h_t\}$ *where*

a) g *is a predictable process.*

b) For each ω, t, $h_t(\omega, \cdot)$ *is a generalized function in* $(0, +\infty)$.

c) For each T, *the process* $h(T)$ *is predictable.* ■

The intuition behind the previous definition is that g_t is the number of units of the risk-free asset in the portfolio at time t, whereas $h_t(T)\,dT$ is the "number" of bonds with maturities between T and $t + dT$ hold at time t in the same portfolio.

Definition 3 *The value process,* V, *of a portfolio* $\{g, h\}$ *is defined by*

$$V_t = g_t B_t + \int_{T=t}^{\infty} P(t,T)\, h(t,T)\, dT \tag{2}$$

where B_t *is the risk-free asset or bank account process and* $P(t,T)$ *is the price, at time* t, *of a zero-coupon bond maturing at* T. ■

Definition 4 *A portfolio is* ***self-financing*** *if its value process satisfies*

$$dV_t = g_t dB_t + \int_{T=t}^{\infty} h(t,T)\, dP(t,T)\, dT$$

■

The previous definition is only formal by now and will be meaningful below.

Applying Itô's rule to the process $\overline{V}_t = V_t B_t^{-1}$ we obtain the following result that allows us to write the self-financing condition in terms of discounted processes.

Lemma 1 *The following conditions are equivalent:*

i) $dV_t = g_t dB_t + \int_{T=t}^{\infty} h(t,T)\, dP(t,T)\, dT$.

ii) $d\overline{V}_t = \int_{T=t}^{\infty} h(t,T)\, d\overline{P}(t,T)\, dT$, *where* $\overline{V}_t$ *and* $\overline{P}(t,T)$ *are the respective processes discounted with respect to* B_t. ■

3 Completeness

In this section we will study market completeness under the stochastic string model of Bueno-Guerrero *et al.* (2014a)

Definition 5 *Consider a discounted contingent claim $\overline{X} \in L^{\infty}(\mathcal{F}_{T_0})$. We say that $\overline{X}$ can be* ***replicated*** *or that we can* ***hedge*** *against $\overline{X}$ if there exists a self-financing portfolio with bounded, discounted value process $\overline{V}$, such that $\overline{V}_{T_0} = \overline{X}$. If every $\overline{X} \in L^{\infty}(\mathcal{F}_{T_0})$ can be replicated, for every T_0, the market is said to be* ***complete****.* ∎

By Lemma 1, the hedging problem for $\overline{X}$ is reduced to find, for each t, a generalized function $h_t(\cdot)$ such that

$$d\overline{V}_t = \int_{T=t}^{\infty} h(t,T)\, d\overline{P}(t,T)\, dT \equiv \varphi_t\left(\frac{d\overline{P}_t}{\overline{P}_t}\right) \tag{3}$$

and

$$\overline{V}_{T_0} = \overline{X} \tag{4}$$

Before solving this problem, we need the following assumption that contains the regularity conditions of the bond price process.

Assumption 1 *The discounted bond return as a function of maturity belongs to the space $\mathcal{D}(t, +\infty)$.*
∎

Bueno-Guerrero *et al.* (2014a) show (Proof of Theorem 9) that the dynamics of $\overline{P}(t,T)$ is given by

$$d\overline{P}(t,T) = -\overline{P}(t,T) \int_{y=0}^{T-t} d\widetilde{Z}(t,y) dy \sigma(t,y) \tag{5}$$

This allows us to write (3) in the more compact form

$$d\overline{V}_t = \varphi_t\left[\Gamma_t\left(d\widetilde{Z}_t\right)\right] \tag{6}$$

where the operator Γ_t is given by

$$\Gamma_t(g_t) = -\int_{y=0}^{\cdot-t} dy g(t,y)\, \sigma(t,y) \tag{7}$$

On the other hand, the portfolio value process is given by $V_t = \mathbb{E}^{\mathbb{Q}}\left\{V_{T_0} e^{-\int_t^{T_0} ds r(s)} \,|\mathcal{F}_t\right\}$, from where $\overline{V}_t = \mathbb{E}^{\mathbb{Q}}\left\{\overline{X}\,|\mathcal{F}_t\right\}$, that is a martingale under the equivalent martingale measure $\mathbb{Q}$. By the martingale representation assumption in Bueno-Guerrero *et al.* (2014a) (Assumption 4.6), we have that

$$d\overline{V}_t = \int_{u=0}^{\infty} d\widetilde{Z}(t,u)\, du j(t,u) \tag{8}$$

with $j(t,u)$ an adapted predictable process for each u.

Joining (6) and (8) we have

$$\varphi_t\left[\Gamma_t\left(d\widetilde{Z}_t\right)\right]=\int_{y=0}^{\infty}d\widetilde{Z}(t,y)\,dyj(t,y) \tag{9}$$

that can be rewritten as

$$\varphi_t\left[\Gamma_t\left(d\widetilde{Z}_t\right)\right]=\int_{y=0}^{\infty}\left[\Gamma_t^{-1}\left(\Gamma_t\left(d\widetilde{Z}_t\right)\right)\right](y)\,dyj(t,y) \tag{10}$$

From the definition of the operator Γ_t, expression (7), it is easy to show that its inverse, Γ_t^{-1}, is given by

$$\left[\Gamma_t^{-1}(q_t)\right](y)=-\frac{1}{\sigma(t,y)}q_t'(t+y)\quad\forall q_t\in\operatorname{Im}\Gamma_t$$

that makes sense because $\sigma>0$ in the stochastic string modeling. Substituting this expression in (10) and using (7), we have

$$\begin{aligned}\varphi_t\left(\frac{d\overline{P}_t}{\overline{P}_t}\right)&=-\int_{y=0}^{\infty}\left(\frac{d\overline{P}_t}{\overline{P}_t}\right)'(t+y)\,\frac{j(t,y)}{\sigma(t,y)}dy\\&=-\int_{T=t}^{\infty}\left(\frac{d\overline{P}_t}{\overline{P}_t}\right)'(T)\,\frac{j(t,T-t)}{\sigma(t,T-t)}dT\\&=\int_{T=t}^{\infty}\left(\frac{d\overline{P}_t}{\overline{P}_t}\right)(T)\left[\frac{j(t,T-t)}{\sigma(t,T-t)}\right]'dT\end{aligned}$$

where in the last equality we have taken into account Assumption 1 and the definition of the generalized derivative. Comparing with expression (3) we arrive at the following result.

Theorem 1 *In the stochastic string model of Bueno-Guerrero* et al. *(2014a), the generalized function $h_t(\cdot)$, solution of (3)-(4), is given by*

$$h(t,T)=\frac{1}{\overline{P}(t,T)}\left[\frac{j(t,T-t)}{\sigma(t,T-t)}\right]' \tag{11}$$

where the symbol $'$ means derivative with respect to T in the sense of distributions and $j(t,\cdot)$ is given by the martingale representation of $\overline{V}_t$

$$d\overline{V}_t=\int_{u=0}^{\infty}d\widetilde{Z}(t,u)duj(t,u) \tag{12}$$

and $\widetilde{Z}(t,u)$ is the stochastic string process with respect to the equivalent martingale measure. ∎

Corollary 1 *In the stochastic string framework of Bueno-Guerrero* et al. *(2014a) the market is complete.* ∎

3.1 Completeness of infinite-dimensional HJM models

We have just obtained market completeness within the stochastic string modeling. Bueno-Guerrero *et al.* (2014b) showed that, under some conditions, infinite-dimensional HJM models are particular cases of the stochastic string framework. So this type of models must verify market completeness too. Nevertheless, in a recent paper, Barski *et al.* (2011) show that the market is not complete in infinite-dimensional HJM models. In this section, we study this apparent contradiction.

Barski *et al.* (2011) work with an infinite-dimensional HJM model given by the forward curve dynamics

$$df\left(t,T\right)=\alpha\left(t,T\right)dt+\sum_{i=1}^{\infty}\sigma^{i}\left(t,T\right)dW^{i}(t)$$

where $W(t)=\left(W^{1}(t),W^{2}(t),\ldots\right)$ is a cylindrical Wiener process in l^{2}. These authors demonstrate that $\overline{P}$ is a martingale with values in $G=H^{1}\left[0,\widehat{T}\right]$, and, consequently, they take admissible strategies as G^{*}-valued processes. Their main result of market incompleteness is obtained by proving that there exists bounded random variables that cannot be replicated with admissible strategies. They also show that the market is complete if we enlarge the class of admissible strategies to processes stochastically integrable with respect to $\overline{P}$, and if the operator $\Gamma_{t}^{BJZ}:l^{2}\rightarrow G$ given by

$$\left(\Gamma_{t}^{BJZ}u\right)(T)=-\overline{P}\left(t,T\right)\sum_{i=1}^{\infty}\left[\int_{y=0}^{T}dy\sigma^{i}\left(t,y\right)\right]u^{i},\quad u\in l^{2}$$

is injective. The operator Γ_{t}^{BJZ} verifies $d\overline{P}\left(t,T\right)=\left(\Gamma_{t}^{BJZ}dW_{t}\right)(T)$, i.e., it maps the infinite-dimensional stochastic shock into the discounted price change, thus, it plays a role in infinite-dimensional HJM models similar to the role played by the operator Γ_{t} in our stochastic string model. The main difference between the two approaches is that it is not possible in general to invert the operator Γ_{t}^{BJZ} as we did with our Γ_{t}. Moreover, it is easy to see that a sufficient condition for $Ker\,\Gamma_{t}^{BJZ}=\{0\}$ is that $\left\{\sigma_{t}^{i}\right\}_{i=0}^{\infty}$ is a linearly independent set. Barski *et al.* (2011) take advantage of this to give an example of model with orthogonal (and consequently linearly independent) volatilities that is complete. We will see below that orthogonality is the key property in infinite-dimensional HJM models to obtain completeness.

Before stating the completeness result we need to review some concepts from Bueno-Guerrero *et al.* (2014b). They show that the infinite-dimensional process $Z_{P}(t,x)$ defined by

$$dZ_{P}\left(t,x\right)=\sum_{i=0}^{\infty}\frac{\sigma_{HJM}^{(i)}(t,x)}{\sigma(t,x)}dW_{i}(t)\tag{13}$$

is a stochastic string shock if some regularity conditions over the $\sigma_{HJM}^{(i)}(t,x)$ are fulfilled. With this process, a stochastic string model with volatility $\sigma(t,x)$ is transformed into an infinite-dimensional

HJM model with volatilities $\sigma^{(i)}_{HJM}(t,x)$.[2] It is also proved that the volatilities can be written as $\sigma^{(i)}_{HJM}(t,x) = \sqrt{\lambda_{t,i}} f_{t,i}(x)$ where $\{\lambda_{t,i}\}_{i=0}^{\infty}$ and $\{f_{t,i}\}_{i=0}^{\infty}$ are, respectively, the eigenvalues and the corresponding orthonormal eigenfunctions of the Hilbert-Schmidt integral operator L_{R_t} defined in $L^2(\mathbb{R}^+)$ by

$$L_{R_t} f(x) = \int_{y=0}^{\infty} dy R_t(x,y) f(y) dy$$

where $R_t(x,y)$ is the instantaneous conditional covariance between shocks to the forward curve.[3] Moreover, the volatilities are orthogonal in $L^2(\mathbb{R}^+)$ verifying $\left\langle \sigma^{(i)}_{HJM,t}, \sigma^{(j)}_{HJM,t} \right\rangle_{L^2(\mathbb{R}^+)} = \lambda_{t,i}\delta_{i,j}$.

We can now state the following result.

Theorem 2 *Under infinite-dimensional HJM models obtained within the stochastic string framework, the market is complete. Moreover, if the operators L_{R_t} are injective, then the hedging strategy is given by*

$$h(t,T) = \frac{1}{\overline{P}(t,T)} \sum_{i=0}^{\infty} \frac{j_i(t)}{\lambda_{t,i}} \overline{\sigma}'^{(i)}_{HJM}(t,T) \tag{14}$$

where $\lambda_{t,i}$ are the eigenvalues of L_{R_t}, $\overline{\sigma}^{(i)}_{HJM}(t,T)$ are the HJM volatilities in the maturity parameterization, $j(t) = (j_0(t), j_1(t), \ldots)$ is a predictable l^2-valued process given by the martingale representation

$$d\overline{V}_t = \left\langle j(t), d\widetilde{W}(t) \right\rangle_{l^2}$$

and $\widetilde{W}(t)$ is cylindrical Wiener process with respect to the equivalent martingale measure. ∎

Proof: See the Appendix. ∎

4 Hedging contingent claims

In this section we apply Theorem 1 to obtain a hedging portfolio for a very general type of European options. The main result is the following.

Proposition 1 *The contingent claim determined by*

$$X_{T_0} = [\Phi(\mathbf{P}_{T_0})]_+ \equiv [\Phi(P(T_0,T_0), \ldots, P(T_0,T_n))]_+$$

with $\Phi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ an homogeneous function of degree one, can be replicated by a portfolio composed by $n_i(t)$ bonds with maturities T_i, $i = 0, \ldots, n$, with

$$n_i(t) = \mathbb{E}^{\mathbb{Q}_{T_i}} \left\{ \left. \frac{\partial \Phi(\mathbf{y})}{\partial y_i} \right|_{\mathbf{y}=\mathbf{P}_{T_0}} \mathbf{1}_{\Phi(\mathbf{P}_{T_0})>0} \middle| \mathcal{F}_t \right\}, \quad t \le T_0 \tag{15}$$

[2]It is understood that we work in the Musiela parameterization.

[3]Bueno-Guerrero *et al.* (2014b) actually work with $L^2_p(\mathbb{R}^+)$ for consistency purposes. It is not difficult to check that all in this paper could be done working with $L^2(\mathbb{R}^+)$.

■

Proof: See the Appendix. ■

Corollary 2 *Under the conditions of Proposition 1, if* $\left.\frac{\partial\Phi(\mathbf{y})}{\partial y_i}\right|_{\mathbf{y}=\mathbf{P}_{T_0}}$ *is* $\mathcal{F}_t$*-measurable, then*

$$n_i(t) = \left.\frac{\partial\Phi(\mathbf{y})}{\partial y_i}\right|_{\mathbf{y}=\mathbf{P}_{T_0}} \mathbb{Q}_{T_i}\left\{\Phi(\mathbf{P}_{T_0}) > 0 \middle| \mathcal{F}_t\right\}, \qquad t \le T_0$$

i.e., the number of bonds with maturity T_i *in the hedging portfolio is proportional to the conditional probability under the* T_i*-forward measure of the event that the derivative ends up in the money.* ■

Note that usually the measurability condition on the partial derivative is satisfied when $\left.\frac{\partial\Phi(\mathbf{y})}{\partial y_i}\right|_{\mathbf{y}=\mathbf{P}_{T_0}}$ is a constant independent of $\mathbf{P}_{T_0}$.

In the Gaussian case it is possible to obtain a closed formula for $n_j(t)$ as is shown in the following result.

Proposition 2 *If* $\sigma(t,x)$ *and* $c(t,x,y)$ *are deterministic, then*

$$n_j(t) = \int_{I_t} d^N x g\left(x_1 - \frac{\Delta_{1i}}{\sqrt{\Delta_{11}}}, \cdots, x_n - \frac{\Delta_{ni}}{\sqrt{\Delta_{nn}}}; M\right) \left.\frac{\partial\Phi(\mathbf{y})}{\partial y_j}\right|_{\mathbf{y}=\mathbf{P}_{T_0}(t,\mathbf{x})}$$

where

$$g(x_1,\ldots,x_N;M) = \frac{1}{\sqrt{(2\pi)^N |M|}} \exp\left(-\frac{1}{2}\sum_{i,j=1}^{N} x_i \left(M^{-1}\right)_{ij} x_j\right)$$

is the density function of a multivariate normal random variable, M *is the correlation matrix given by* $(M)_{kl} = \frac{\Delta_{kl}}{\sqrt{\Delta_{kk}}\sqrt{\Delta_{ll}}}$*,* $k,l = 1,\ldots,N$ *with*

$$\Delta_{ij}(t,T_0) = \int_{s=t}^{T_0} ds \left[\int_{y=T_0-s}^{T_i-s}\int_{u=T_0-s}^{T_j-s} dy du R_s(u,y)\right]$$

and

$$\mathbf{P}_{T_0}(t,\mathbf{x}) = \left(1, e^{\sqrt{\Delta_{11}}x_1 - \frac{1}{2}\Delta_{11}}\frac{P(t,T_1)}{P(t,T_0)}, \cdots, e^{\sqrt{\Delta_{NN}}x_N - \frac{1}{2}\Delta_{NN}}\frac{P(t,T_N)}{P(t,T_0)}\right)$$

$$I_t = \left\{\mathbf{x} \in \mathbb{R}^n : \Phi(\mathbf{P}_{T_0}(t,\mathbf{x})) > 0\right\}$$

■

Proof: See the Appendix. ■

Corollary 3 *Under the conditions of Corollary 2 and Proposition 2 it is verified that*

$$n_j(t) = \left.\frac{\partial \Phi(\mathbf{y})}{\partial y_j}\right|_{\mathbf{y}=\mathbf{P}_{T_0}} \int_{I_t} d^N x g\left(x_1 - \frac{\Delta_{1i}}{\sqrt{\Delta_{11}}}, \cdots, x_n - \frac{\Delta_{ni}}{\sqrt{\Delta_{nn}}}; M\right)$$

and in this case

$$\mathbb{Q}_{T_i}\left\{\Phi\left(\mathbf{P}_{T_0}\right) > 0 \middle| \mathcal{F}_t\right\} = \int_{I_t} d^N x g\left(x_1 - \frac{\Delta_{1i}}{\sqrt{\Delta_{11}}}, \cdots, x_n - \frac{\Delta_{ni}}{\sqrt{\Delta_{nn}}}; M\right)$$

■

Some derivatives belonging to the class treated in this section are already priced under the stochastic string Gaussian model. For example, call options in Bueno-Guerrero *et al.* (2014a, b) and caps and swaptions in Bueno-Guerrero *et al.* (2014c). Concretely, the price at time t of a European call option that matures at time T_0 with strike K written on a zero-coupon bond that matures at time $T > T_0$ is given by

$$\mathbf{Call}_K\left[t, T_0, T\right] = P\left(t, T\right)\mathcal{N}\left[d_1\left(t, T_0, T\right)\right] - KP\left(t, T_0\right)\mathcal{N}\left[d_2\left(t, T_0, T\right)\right]$$

where $\mathcal{N}(\cdot)$ denotes the distribution function of a standard normal random variable with

$$d_1\left(t, T_0, T\right) = \frac{\ln\left(\frac{P\left(t, T\right)}{KP\left(t, T_0\right)}\right) + \frac{1}{2}\Omega\left(t, T_0, T\right)}{\sqrt{\Omega\left(t, T_0, T\right)}}, \quad d_2\left(t, T_0, T\right) = d_1 - \sqrt{\Omega\left(t, T_0, T\right)}$$

and

$$\Omega\left(t, T_0, T\right) = \int_{v=t}^{T_0} dv\left[\int_{y=T_0-v}^{T-v}\int_{w=T_0-v}^{T-v} dydwR_v\left(u, y\right)\right]$$

To end this section we will hedge (and price) a compound option, another derivative with homogeneous payoff function of degree one. For illustration purposes, the next example shows the computations for a call on a call.[4]

Example 1 *Consider a European call option with maturity T_0 and strike K_C on another European call option maturing at $T_1 > T_0$ and strike K written on a zero-coupon bond with maturity $T_2 > T_1$. Let $\mathbf{CoC}\left(t, T_0, T_1, T_2\right)$ denote the price at time t of this option. Given its payoff we have*

$$\begin{aligned}\Phi\left[P\left(T_0, T_0\right), P\left(T_0, T_1\right), P\left(T_0, T_2\right)\right] &= \mathbf{Call}_K\left[T_0, T_1, T_2\right] - K_C P\left(T_0, T_0\right) \\ &= P\left(T_0, T_2\right)\mathcal{N}\left[d_1\left(T_0, T_1, T_2\right)\right] - KP\left(T_0, T_1\right)\mathcal{N}\left[d_2\left(T_0, T_1, T_2\right)\right] \\ &- K_C P\left(T_0, T_0\right)\end{aligned}$$

[4]Other types of compound options can be priced and hedged in a similar way.

from where we obtain

$$\left.\frac{\partial\Phi\left(\mathbf{y}\right)}{\partial y_0}\right|_{\mathbf{y}=\mathbf{P}_{T_0}}=-K_C,\quad \left.\frac{\partial\Phi\left(\mathbf{y}\right)}{\partial y_1}\right|_{\mathbf{y}=\mathbf{P}_{T_0}}=-K\mathcal{N}\left[d_2\left(T_0,T_1,T_2\right)\right],\quad \left.\frac{\partial\Phi\left(\mathbf{y}\right)}{\partial y_2}\right|_{\mathbf{y}=\mathbf{P}_{T_0}}=\mathcal{N}\left[d_1\left(T_0,T_1,T_2\right)\right]$$

So by Proposition 1 and Corollary 3 we obtain

$$\mathbf{CoC}\left(t,T_0,T_1,T_2\right)=\sum_{i=0}^{2}n_i\left(t\right)P\left(t,T_i\right)$$

with

$$\begin{aligned} n_0(t)&=-K_C\int_{-\infty}^{+\infty}dx_1\int_{l_t}^{+\infty}dx_2 g\left(x_1,x_2;M\right)\\ n_1(t)&=-K\mathcal{N}\left[d_2\left(T_0,T_1,T_2\right)\right]\int_{-\infty}^{+\infty}dx_1\int_{l_t}^{+\infty}dx_2 g\left(x_1-\sqrt{\Delta_{11}},x_2-\frac{\Delta_{21}}{\sqrt{\Delta_{22}}};M\right)\\ n_2(t)&=\mathcal{N}\left[d_1\left(T_0,T_1,T_2\right)\right]\int_{-\infty}^{+\infty}dx_1\int_{l_t}^{+\infty}dx_2 g\left(x_1-\frac{\Delta_{12}}{\sqrt{\Delta_{11}}},x_2-\sqrt{\Delta_{22}};M\right) \end{aligned}$$

and

$$l_t=\frac{1}{\sqrt{\Delta_{22}}}\left(\frac{\Delta_{22}}{2}+\ln\left(\frac{K_C P\left(t,T_0\right)+KP\left(t,T_1\right)e^{\sqrt{\Delta_{11}}x_1-\frac{1}{2}\Delta_{11}}\mathcal{N}\left[d_2\left(T_0,T_1,T_2\right)\right]}{P\left(t,T_2\right)\mathcal{N}\left[d_1\left(T_0,T_1,T_2\right)\right]}\right)\right)$$

■

5 Uniqueness of the equivalent martingale measure

In this section we state the conditions under which the equivalent martingale measure is unique in the stochastic string model of Bueno-Guerrero *et al.* (2014a). We review all that we need from this paper.

Bueno-Guerrero *et al.* (2014a) start from the following dynamics for the instantaneous forward interest rate $f(t,x)$ in the Musiela parameterization

$$df(t,x)=\alpha(t,x)dt+\sigma(t,x)dZ(t,x)$$

with $\sigma>0$ and where $Z\left(t,x\right)$ is the stochastic string process. To guarantee the absence of arbitrage, they assume the existence of a martingale measure $\mathbb{Q}$ equivalent to the physical probability $\mathbb{P}$. By defining the martingale

$$\eta_t=\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\,|\mathcal{F}_t\right]$$

they find that it can be expressed as $\eta_t = \varepsilon\left(N\right)_t$, where ε is the stochastic exponential and

$$N_t = -\int_{v=0}^{t}\int_{u=0}^{\infty} dZ(v,u)du\lambda(v,u) \tag{16}$$

being $\lambda\left(v,u\right)$ the *specific market price of risk* associated with time to maturity u.

Finally, they find the relationship between stochastic string shocks under the two probability measures

$$dZ^{\mathbb{Q}}(t,y) = dZ^{\mathbb{P}}(t,y) + dt\int_{u=0}^{\infty} duc(t,u,y)\lambda(t,u)$$

and the no-arbitrage condition

$$\alpha(t,x) = \frac{\partial f(t,x)}{\partial x} + \sigma(t,x)\left[\int_{y=0}^{x} dyc(t,x,y)\sigma(t,y) + \int_{y=0}^{\infty} dyc(t,x,y)\lambda(t,y)\right] \tag{17}$$

where $c(t,x,y)$ is the correlation between stochastic string shocks.

Before addressing the main result of this section, it is interesting to study why the two concepts of completeness and uniqueness of the martingale measure are unrelated in our model, whereas in other papers they are related though duality relations. We will do so by comparing our approach with the ones proposed in BKR and Jarrow and Madam (1999).

BKR define a pair of (families of) adjoint operators called *martingale operators*, $\chi_{BKR,t}$, and *hedging operators*, $\chi^*_{BKR,t}$. They show that $\mathbb{Q}$ is unique if and only if χ_{BKR} is injective and that the market is complete if and only if χ^*_{BKR} is surjective. Using the duality relation $\left(Ker\,\chi\right)^{\perp} = cl\ \left(\mathrm{Im}\,\chi^*\right)$ they also obtain the equivalence between uniqueness of the martingale measure and approximate completeness.

In our model, replacing (8) and (5) in (3) we have

$$-\int_{T=t}^{\infty} h\left(t,T\right)\left[\overline{P}\left(t,T\right)\int_{y=0}^{T-t} d\widetilde{Z}(t,y)dy\sigma(t,y)\right]dT = \int_{y=0}^{\infty} d\widetilde{Z}(t,y)dyj(t,y)$$

that can be written as

$$-\int_{y=0}^{\infty} d\widetilde{Z}(t,y)dy\sigma(t,y)\left[\int_{T=t+y}^{\infty} h\left(t,T\right)\overline{P}\left(t,T\right)dT\right] = \int_{y=0}^{\infty} d\widetilde{Z}(t,y)dyj(t,y)$$

from where we obtain

$$-\sigma(t,y)\int_{T=t+y}^{\infty} h\left(t,T\right)\overline{P}\left(t,T\right)dT = j(t,y)$$

So, we can define the operator

$$\begin{aligned} \chi^*_t &: \mathcal{D}'\left(0,+\infty\right) \longrightarrow \mathcal{D}'\left(0,+\infty\right) \\ m_t &\longmapsto -\sigma\left(t,\cdot\right)\int_{T=t+\cdot}^{\widehat{T}} m\left(t,T\right)\overline{P}\left(t,T\right)dT \end{aligned}$$

that is the analogous to the hedging operator in our framework. We know from the completeness of our model, Theorem 1, that χ_t^* is suprajective.

The operator χ_t^* is the adjoint of the operator

$$\begin{aligned}\chi_t &: \mathcal{D}\left(]0,+\infty[\right) \longrightarrow \mathcal{D}\left(]0,+\infty[\right)\\ \Omega_t &\longmapsto -\sigma\left(t,\cdot\right)\overline{P}\left(t,T\right)\Omega\left(t,\cdot\right)\end{aligned}$$

that is the martingale operator in our setting. This operator is clearly injective and we can obtain $\left[\Theta_t\lambda_t\right](x) = \int_{y=0}^{\infty} dyc\left(t,x,y\right)\lambda\left(t,y\right)$ as the unique solution of the no-arbitrage condition, equation (17).

Until now the parallelism with BKR is complete. But, in their paper, the injectiveness of χ_{BKR} allows them to identify uniquely the martingale measure. Nevertheless, in our model, to this end, we need to identify $\lambda\left(t,y\right)$, that can not be obtained uniquely unless Θ_t is injective.

Jarrow and Madam (1999) develop a very general model to study the relationship between market completeness and uniqueness of martingale measures. They define the so called *fundamental drift operator*, T_{JM}, that maps the space of potential measures $\mathbb{M}$ into the space of excess mean returns $\mathbb{X}$, and show that injectiveness of this operator is equivalent to uniqueness of signed local martingale measures. Using the theory of linear operators between locally convex topological vector spaces, they obtain a version of the Second Fundamental Theorem that states the equivalence between uniqueness of signed martingale measures and quasicompleteness, and that if T_{JM} is an open mapping, then uniqueness is equivalent to completeness.

Nevertheless, to take advantage of this approach, we had to work, as in Jarrow and Madam (1999), with time-independent operators acting on spaces of processes. Instead, like BKR or Barski *et al.* (2011), in this paper we work with families of operators (indexed by t) taking values in functional spaces.

From what we have said above, it is clear that the equivalent martingale measure is unique if and only if the equation (17) has a unique solution for $\lambda\left(t,u\right)$. To study the conditions in which this is possible, we define, for each t, the integral operator $\Theta_t : H_t \to H_t$ given by

$$\left[\Theta_t\lambda_t\right](x) = \int_{y=0}^{\infty} dyc\left(t,x,y\right)\lambda\left(t,y\right)$$

where H_t, the set of possible specific market prices of risk, is for each t, a Hilbert subspace of $L^2\left(\mathbb{R}^+\right)$ that we have to find with the condition that Θ_t be injective.

As $c\left(t,x,y\right)$ is a correlation function for each t, Θ_t is a compact self-adjoint operator because it is a Hilbert-Schmidt integral operator with symmetric kernel. This property allows us to state the following result.

Theorem 3 *The equivalent martingale measure is unique if and only if the possible specific market prices of risk can be expressed as Fourier series of the eigenfunctions of* Θ_t. ∎

Proof: See the Appendix. ∎

In our model, the analog to the fundamental operator in Jarrow and Madam (1999) is Θ_t. The equivalent martingale measure is unique if and only if Θ_t is injective and, in this case, Θ_t is also suprajective because $H_t = \operatorname{Im}\Theta_t \oplus Ker\,\Theta_t$. As Θ_t is a bounded linear application between two Banach spaces, it is also open by the Open Mapping Theorem.[5]

6 Conclusions

It is well known that the market is complete under multi-factor HJM models. However, infinite-dimensional HJM models, in general, do not have this property, as shown by Barski *et al.* (2011). This lack of completeness is a characteristic shared by many other models. As a consequence, we might think that the infinite-dimensional modeling of interest rates is not compatible with market completeness. In this paper, we show that this is not the case. We demonstrate that in the infinite-dimensional stochastic string model, the market is complete when we take distribution-valued processes as trading strategies. In our model, these two issues are necessary to obtain completeness. The stochastic string modeling allows us to obtain explicitly the inverse of the operator that maps stochastic shocks into bond returns and the use of distributions is key to identify the hedging strategy. Although it is not the main objective of this paper, we obtain market completeness for infinite-dimensional HJM models within the stochastic string framework, showing the importance of this approach. The orthogonality of HJM volatilities, that is verified in stochastic string models, is the characteristic needed to obtain completeness.

As an application of the completeness result, we provide a hedging strategy for options with payoff functions that are homogeneous of degree one. This strategy simply consists of holding bonds with the same maturities as the bonds underlying the option. This is a property of the infinite-dimensional modeling not shared with multi-factor models in which the maturities can be stated a priori, as noted by Carmona and Tehranchi (2004). We obtain the hedging strategy without the need to use any version of the Clark-Ocone formula of the Malliavin calculus.

Although the uniqueness of the martingale measure is not required by our fundamental result of completeness, we also study uniqueness in the stochastic string framework. We obtain a result of equivalence between uniqueness of the martingale measure and the representation of specific market risk premia as Fourier series of the eigenvalues of certain integral operator similar to the fundamental

[5] Our model fits with the Jarrow and Madam framework by just interchanging their dual pair $(\mathbb{X}, \mathbb{Y})$ by our pair $(\mathcal{D}, \mathcal{D}')$. However, we think that our approach is financially more sound because it uses specific market prices of risk.

operator of Jarrow and Madam (1999). Moreover we show that this property is equivalent to the property that the operator be an open application, a condition necessary for the equivalence between uniqueness and completeness in the version of the Second Fundamental Theorem of these last authors.

References

[1] Barski, M., J. Jakubowski and J. Zabczyk (2011): "On Incompleteness of Bond Markets with Infinite Number of Random Factors", *Mathematical Finance*, 21, 3, 541-556.

[2] Björk, T (2004): *Arbitrage Theory in Continuous Time*, second edition, Oxford University Press.

[3] Björk, T.; G. Di Masi; Y. Kabanov and W. Runggaldier (1997): "Towards a General Theory of Bonds Markets", *Finance and Stochastics*, 1, 141-174.

[4] Björk, T.; Y. Kabanov and W. Runggaldier (2002): "Bond Market Structure in the Presence of Marked Point Processes", *Mathematical Finance*, 7, 2, 211-239.

[5] Bueno-Guerrero, A.; M. Moreno and J.F. Navas (2014a): "Stochastic String Models with Continuous Semimartingales". Available at http://ssrn.com/abstract=2438601.

[6] Bueno-Guerrero, A.; M. Moreno and J.F. Navas (2014b): "The Stochastic String Model as a Unifying Theory of the Term Structure of Interest Rates". Available online at http://ssrn.com/abstract=2438676.

[7] Bueno-Guerrero, A.; M. Moreno and J.F. Navas (2014c): "Valuation of Caps and Swaptions under a Stochastic String Model". Available at http://ssrn.com/abstract=2438678.

[8] Carmona, R. and M. Tehranchi (2004): "A Characterization of Hedging Portfolios for Interest Rate Contingent Claims", *Annals of Applied Probability*, 14, 3, 1267-1294.

[9] De Donno, M. (2004): "The Term Structure of Interes Rates as a Random Field: A Stochastic Integration Approach", in *Stochastic Processes and Applications to Mathematical Finance*, Prodeedings of the Ritsumeikan International Symposium, 27-51, World Sci. Publishing, River Edge, NJ.

[10] De Donno, M. and M. Pratelli (2004): "On the Use of Measure-Valued Strategies in Bond Markets", *Finance and Stochastics*, 8, 87-109.

[11] Harrison, J.M. and S.R. Pliska (1981): "Martingales and Stochastic Integrals in the Theory of Continuous Trading", *Stochastic Processes and their Applications*, 11, 215-260.

[12] Jarrow, R. and Madam (1999): "Hedging Contingent Claims on Semimartingales", *Finance and Stochastics*, 3, 111-134.

[13] Kennedy, D. P. (1994): "The Term Structure of Interest Rates as a Gaussian Random Field", *Mathematical Finance*, 4, 247-258.

[14] Schwartz, L. (1966): *Théorie des Distributions*, Hermann.

[15] Taflin, E. (2005): "Bond Market Completeness and Attainable Contingent Claims", *Finance and Stochastics*, 9, 429-452.

Appendix of Proofs

Proof of Theorem 2

In the infinite-dimensional HJM setting, the equation (9) is written as

$$\varphi_t\left[\widehat{\Gamma}_t^{BJZ}\left(d\widetilde{W}(t)\right)\right] = \left\langle j(t), d\widetilde{W}(t)\right\rangle_{l^2} \tag{18}$$

with $\widehat{\Gamma}_t^{BJZ} \equiv \frac{1}{\overline{P}(t,T)}\Gamma_t^{BJZ}$. Differentiating with respect to T in the equation $\left[\widehat{\Gamma}_t^{BJZ}u\right](T) = 0$ we obtain $\sum_{i=0}^{\infty}\sigma_{HJM}^{(i)}(t,T)\,u_i = 0$. As HJM volatilities are orthogonal within the stochastic string framework, we have that $\left\{\sigma_{HJM,T}^{(i)}\right\}_{i=0}^{\infty}$ is a linearly independent set and thus we have $u = 0$. So $\widehat{\Gamma}_t^{BJZ}$ are injective and we can rewrite (18) as

$$\varphi_t\left[\widehat{\Gamma}_t^{BJZ}\left(d\widetilde{W}(t)\right)\right] = \left\langle j(t), \left[\left(\widehat{\Gamma}_t^{BJZ}\right)^{-1}\widehat{\Gamma}_t^{BJZ}\right]\left(d\widetilde{W}(t)\right)\right\rangle_{l^2}$$

If we take into account that for infinite-dimensional HJM models it is verified that

$$\frac{d\overline{P}_t}{\overline{P}_t} = -\sum_{i=0}^{\infty}\left[\int_{y=0}^{T} dy\,\sigma_{HJM}^{(i)}(t,y)\right]d\widetilde{W}_i(t) = \widehat{\Gamma}_t^{BJZ}\left(d\widetilde{W}(t)\right)$$

we can define the functional φ_t given by

$$\varphi_t[v_t] = \left\langle j(t), \left(\widehat{\Gamma}_t^{BJZ}\right)^{-1}v_t\right\rangle_{l^2}, \quad v_t \in \operatorname{Im}\widehat{\Gamma}_t^{BJZ} \subset \mathcal{D}(0,+\infty)$$

As $\varphi_t \in \mathcal{D}'(0,+\infty)$, there exists a generalized function $h(t,T)$, such that

$$\varphi_t\left(\frac{d\overline{P}_t}{\overline{P}_t}\right) = \int_{T=t}^{\infty} h(t,T)\,d\overline{P}(t,T)\,dT = \left\langle j(t), d\widetilde{W}(t)\right\rangle_{l^2}$$

We have just proved that the market is complete. To obtain the formula (14) we first need to obtain the relationship between martingale representations in the stochastic string and the infinite-dimensional HJM frameworks. Equating both representations of $d\overline{V}_t$ and using equation (13), we obtain

$$\sum_{i=0}^{\infty}\left[\int_{y=0}^{\infty} dy\frac{j(t,y)}{\sigma(t,y)}\sigma_{HJM}^{(i)}(t,y)\right]d\widetilde{W}_i(t) = \sum_{i=0}^{\infty} j_i(t)d\widetilde{W}_i(t)$$

from which we arrive at

$$j_i(t) = \int_{y=0}^{\infty} dy\frac{j(t,y)}{\sigma(t,y)}\sigma_{HJM}^{(i)}(t,y), \quad i = 0,1,\ldots \tag{19}$$

The operator L_{R_t} is a compact self-adjoint operator because it is a Hilbert-Schmidt integral operator with symmetric kernel. By the Spectral Theorem for compact self-adjoint operators, $L^2(\mathbb{R}^+) = \operatorname{Im}L_{R_t} \oplus Ker\,L_{R_t}$ and every $g_t \in L^2(\mathbb{R}^+)$ can be written uniquely as $g_t = \sum_{i=0}^{\infty}(f_{i,t}, g_t)\,f_{i,t} + v_t$

with $\{f_{i,t}\}_{i=0}^{\infty}$ an orthonormal basis of $\operatorname{Im} L_{R_t}$ formed by eigenfunctions of L_{R_t} and $v_t \in Ker\, L_{R_t}$. So, as L_{R_t} is injective, $L^2(\mathbb{R}^+) = \operatorname{Im} L_{R_t}$ and $\{f_{i,t}\}_{i=0}^{\infty} = \left\{ \frac{1}{\sqrt{\lambda_{t,i}}} \sigma_{HJM,t}^{(i)} \right\}_{i=0}^{\infty}$ is an orthonormal basis of $L^2(\mathbb{R}^+)$. Rewriting (19) as

$$\frac{j_i(t)}{\sqrt{\lambda_{t,i}}} = \left\langle \frac{j_t}{\sigma_t}, \frac{\sigma_{HJM,t}^{(i)}}{\sqrt{\lambda_{t,i}}} \right\rangle_{L^2(\mathbb{R}^+)}, \quad i = 0, 1, \dots$$

we can obtain

$$\frac{j_t}{\sigma_t} = \sum_{i=0}^{\infty} \frac{j_i(t)}{\lambda_{t,i}} \sigma_{HJM,t}^{(i)}$$

and, using (11), we arrive at

$$h(t,T) = \frac{1}{\overline{P}(t,T)} \left[\sum_{i=0}^{\infty} \frac{j_i(t)}{\lambda_{t,i}} \sigma_{HJM,t}^{(i)}(T-t) \right]' = \frac{1}{\overline{P}(t,T)} \sum_{i=0}^{\infty} \frac{j_i(t)}{\lambda_{t,i}} \overline{\sigma}_{HJM,t}^{\prime(i)}(t,T)$$

■

Proof of Proposition 1

Under the conditions of the proposition, Bueno-Guerrero *et al.* (2014b) provide the following equality

$$\mathbb{E}^{\mathbb{Q}} \left\{ e^{-\int_{s=t}^{T_0} dt r(s)} \left[\Phi(\mathbf{P}_{T_0}) \right]_+ \Big| \mathcal{F}_t \right\} = \sum_{i=0}^{N} P(t,T_i) \, \mathbb{E}^{\mathbb{Q}_{T_i}} \left\{ \left. \frac{\partial \Phi(\mathbf{y})}{\partial y_i} \right|_{\mathbf{y}=\mathbf{P}_{T_0}} \mathbf{1}_{\Phi(\mathbf{P}_{T_0})>0} \Big| \mathcal{F}_t \right\} \tag{20}$$

and dividing by $B(t)$ we arrive at

$$\overline{V}_t = \sum_{i=0}^{n} n_i(t) \, \overline{P}(t,T_i) \tag{21}$$

By its definition, $n_i(t)$ is a martingale under $\mathbb{Q}_{T_i}$ and by the martingale representation assumption, we can write

$$n_i(t) = n_i(0) + \int_{s=0}^{t} \int_{y=0}^{\infty} d\widetilde{Z}^{T_i}(s,y) \, dy g_i(s,y), \quad t \leq T_0$$

with $g_i(s,y)$ undetermined. Passing to the equivalent martingale measure we have

$$n_i(t) = n_i(0) + \int_{s=0}^{t} \int_{y=0}^{\infty} \int_{x=0}^{T_i - s} ds dy dx c(s,x,y) \, \sigma(s,x) \, g_i(s,y) + \int_{s=0}^{t} \int_{y=0}^{\infty} d\widetilde{Z}(s,y) \, dy g_i(s,y)$$

or in differential form

$$dn_i(t) = \int_{y=0}^{\infty} \int_{x=0}^{T_i - t} dy dx c(t,x,y) \, \sigma(t,x) \, g_i(t,y) \, dt + \int_{y=0}^{\infty} d\widetilde{Z}(t,y) \, dy g_i(t,y)$$

Applying the product rule and reducing we have

$$n_i(t)\overline{P}(t,T_i) = \int_{s=0}^{t} n_i(s)d\overline{P}(s,T_i) + \int_{s=0}^{t} \overline{P}(s,T_i)\, dn_i(s) + \left[\overline{P}(\cdot,T_i), n_i(\cdot)\right]_t$$
$$= \int_{s=0}^{t} \overline{P}(s,T_i) \int_{y=0}^{\infty} d\widetilde{Z}(s,y)dy \left[g_i(s,y) - n_i(s)\,\sigma(s,y)\mathbf{1}_{y<T_i-s}\right]$$

from which we obtain the expression

$$\overline{V}_t = \int_{s=0}^{t}\int_{y=0}^{\infty} d\widetilde{Z}(s,y)\, dy \left[\sum_{i=0}^{n} g_i(s,y)\,\overline{P}(s,T_i) - \sigma(s,y)\sum_{i=0}^{n} n_i(s)\,\overline{P}(s,T_i)\,\mathbf{1}_{y<T_i-s}\right]$$

that allows us to identify $j(t,y)$ in the martingale representation (12) as

$$j(t,y) = \sum_{i=0}^{n} g_i(t,y)\,\overline{P}(t,T_i) - \sigma(t,y)\sum_{i=0}^{n} n_i(t)\,\overline{P}(t,T_i)\,\mathbf{1}_{y<T_i-t}$$

Substituting the value of $j(t,y)$ given by the previous expression in (11) and using that $(\mathbf{1}_{T<T_i})' = -\delta(T-T_i)$ we obtain

$$h(t,T) = \frac{1}{\overline{P}(t,T)}\sum_{i=0}^{n}\left[\frac{g_i(t,T-t)}{\sigma(t,T-t)}\right]'\overline{P}(t,T_i) + \sum_{i=0}^{n} n_i(t)\,\delta(T-T_i) \tag{22}$$

The value process of the hedging portfolio (self-financing by Theorem 1) is obtained by substituting (22) in (2) to get

$$V_t = g_t B_t + \sum_{i=0}^{n}\left.\frac{g_i(t,T-t)}{\sigma(t,T-t)}\right]_{T=t}^{\infty} P(t,T_i) + \sum_{i=0}^{n} n_i(t)\,P(t,T_i)$$

or, dividing by B_t

$$\overline{V}_t = g_t + \sum_{i=0}^{n}\left.\frac{g_i(t,T-t)}{\sigma(t,T-t)}\right]_{T=t}^{\infty}\overline{P}(t,T_i) + \sum_{i=0}^{n} n_i(t)\,\overline{P}(t,T_i)$$

If we compare this expression with (21), we have that

$$g_t + \sum_{i=0}^{n}\left.\frac{g_i(t,T-t)}{\sigma(t,T-t)}\right]_{T=t}^{\infty}\overline{P}(t,T_i) = 0$$

so that the first term in the right hand side of (22) eliminates the bank account part of the hedging portfolio. ∎

Proof of Proposition 2

Following Bueno-Guerrero *et al.* (2014b), we define for $i = 1, \ldots, N,\ j = 0, 1, \ldots, N$, the variables

$$x'_{ij}(t, T_0) \equiv \frac{\ln P(T_0, T_i) - \ln \dfrac{P(t, T_i)}{P(t, T_0)} - \Delta_{ij}(t, T_0) + \dfrac{1}{2}\Delta_{ii}(t, T_0)}{\sqrt{\Delta_{ii}(t, T_0)}}$$

that, under $\mathbb{Q}_{T_j}$, $x'_{ij}(s, T_0)$ have a conditional standard normal distribution. Writing the expectation in (15) in terms of the new variables x'_{ij} we have

$$n_j(t) = \int_{-\infty}^{+\infty} d^N x'_j g\left(x'_{1j}, \ldots, x'_{Nj}; M\right) \left.\frac{\partial \Phi(\mathbf{y})}{\partial y_j}\right|_{\mathbf{y}=\mathbf{P}_{T_0}} \mathbf{1}_{\Phi\left(\mathbf{P}_{T_0}\right)>0}$$

where M is the correlation matrix given by $(M)_{kl} = corr\left(x'_{kj}, x'_{lj}\right) = \dfrac{\Delta_{kl}}{\sqrt{\Delta_{kk}}\sqrt{\Delta_{ll}}},\ k, l = 1, \ldots, N$ and $\Phi(\mathbf{P}_{T_0})$ is explicitly expressed in terms of x'_{ij} by

$$\Phi(\mathbf{P}_{T_0}) = \Phi\left[1, e^{\sqrt{\Delta_{11}}x'_{1j} + \Delta_{1j} - \frac{1}{2}\Delta_{11}} \frac{P(t, T_1)}{P(t, T_0)}, \cdots, e^{\sqrt{\Delta_{NN}}x'_{Nj} + \Delta_{Nj} - \frac{1}{2}\Delta_{NN}} \frac{P(t, T_N)}{P(t, T_0)}\right]$$

Making the change of variable $x_i \equiv x'_{ij} + \dfrac{\Delta_{ij}}{\sqrt{\Delta_{ii}}},\ i = 1, \ldots, N$, we have

$$n_j(t) = \int_{-\infty}^{+\infty} d^N x g(x_1, \ldots, x_N; M)\, e^{\sqrt{\Delta_{jj}}x_j - \frac{1}{2}\Delta_{jj}} \left.\frac{\partial \Phi(\mathbf{y})}{\partial y_j}\right|_{\mathbf{y}=\mathbf{P}_{T_0}} \mathbf{1}_{\Phi\left(\mathbf{P}_{T_0}\right)>0}$$

and applying the identity $g(x_1, \ldots, x_n; M)\, e^{\sqrt{\Delta_{ii}}x_i - \frac{1}{2}\Delta_{ii}} = g\left(x_1 - \dfrac{\Delta_{1i}}{\sqrt{\Delta_{11}}}, \cdots, x_n - \dfrac{\Delta_{ni}}{\sqrt{\Delta_{nn}}}; M\right)$, we finally obtain

$$n_j(t) = \int_{I_t} d^N x g\left(x_1 - \frac{\Delta_{1i}}{\sqrt{\Delta_{11}}}, \cdots, x_n - \frac{\Delta_{ni}}{\sqrt{\Delta_{nn}}}; M\right) \left.\frac{\partial \Phi(\mathbf{y})}{\partial y_j}\right|_{\mathbf{y}=\mathbf{P}_{T_0}}$$

with

$$\mathbf{P}_{T_0}(t, \mathbf{x}) = \left(1, e^{\sqrt{\Delta_{11}}x_1 - \frac{1}{2}\Delta_{11}} \frac{P(t, T_1)}{P(t, T_0)}, \cdots, e^{\sqrt{\Delta_{NN}}x_N - \frac{1}{2}\Delta_{NN}} \frac{P(t, T_N)}{P(t, T_0)}\right)$$
$$I_t = \left\{\mathbf{x} \in \mathbb{R}^n : \Phi\left(\mathbf{P}_{T_0}(t, \mathbf{x})\right) > 0\right\}$$

■

Proof of Theorem 3

The equivalent martingale measure is unique if and only if Θ_t is injective for each t. As in the proof of Theorem 2, Θ_t is injective if and only if $H_t = \operatorname{Im}\Theta_t = \left\{\varkappa_t \in L^2(\mathbb{R}^+) : \varkappa_t = \sum_{i=0}^{\infty}(f_{i,t}, \varkappa_t)\, f_{i,t}\right\}$. ■