Empirical Evaluation of Overspecified Asset Pricing Models*

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Abstract

We study the testing of linear factor pricing models and the estimation of risk prices in potentially overspecified contexts in which we may only estimate a set of risk prices compatible with the pricing conditions. We use single-step GMM procedures, which yield identical inferences with SDF and regression methods, uncentred or centred moments, and symmetric or asymmetric normalizations. We also propose tests to detect problematic cases such as trivial SDFs unrelated to the cross-section of returns. We provide extensive Monte Carlo evidence on estimators and tests. Finally, we apply our methods to some of the most popular asset pricing models.

Keywords: CU-GMM, Factor pricing models, Underidentification tests, Set Estimation, Stochastic discount factor.

JEL: G12, G15, C12, C13.

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1 Introduction

The most popular empirically oriented asset pricing models effectively assume the existence of a common stochastic discount factor (SDF) that is linear in some risk factors, which discounts uncertain payoffs differently across different states of the world. Those factors can be either the excess returns on some traded securities, non-traded economy wide sources of uncertainty related to macroeconomic variables, or a combination of the two. The empirical success of such models at explaining the so called CAPM anomalies was initially limited, but researchers have progressively entertained a broader and broader set of factors, which has resulted in several success claims. Harvey, Liu and Zhu (2014) contains a comprehensive and up to date list of references, cataloguing 315(!) different factors.

However, several authors have warned that some of those factors, or more generally linear combinations of those factors, could be uncorrelated with the vector of excess returns that they are meant to price, which would render them redundant (see Burnside (2014), Gospodinov, Khan and Robotti (2014) and the references therein). Further, those papers forcefully argue that such redundancies can lead to misleading econometric conclusions.

In this context, the purpose of our paper is to study the estimation of risk prices and the testing of the cross-sectional restrictions imposed by overspecified linear factor pricing models in which the candidate SDF depends on too many factors. Our point of departure from the existing literature is that we do not focus exclusively on the properties of the usual estimators and tests. Instead, we use the econometric framework in Arellano, Hansen and Sentana (2012).[1] Specifically, we suppose that under the null hypothesis we may identify a linear subspace of risk prices compatible with the cross-sectional asset pricing restrictions. This set, which can be easily parameterized and efficiently estimated using a standard GMM approach, is of direct interest because it isolates the dimension along which identification of the original linear factor pricing model is problematic. The familiar J test from the work of Sargan (1958) and Hansen (1982) for overidentification of the augmented model now becomes a test for "underidentification" of the original model. For that reason, we refer to it as an I test. If we can identify a linear subspace of risk prices without statistical rejection, then the original asset pricing model is not well identified and we refer to this phenomenon as underidentification. In contrast, a statistical rejection provides evidence that the prices of risk in the original model are indeed point identified, unless of course the familiar J test continues to reject its over-identifying restrictions.

We also follow Peñaranda and Sentana (2014) in using single-step procedures, such as the continuously updated GMM estimator (CU-GMM) of Hansen, Heaton and Yaron (1996), to ob-

[1] In this sense, our paper can be regarded as a substantial extension of Manresa (2009).

tain numerically identical test statistics and risk price estimates for SDF and regression methods, with uncentred or centred moments and symmetric or asymmetric normalizations. In addition, we propose simple tests that can diagnose economically unattractive but empirically relevant cases in which the expected value of the SDF is zero. For simplicity of exposition, we focus on excess returns, but we could easily extend our analysis to cover gross returns too, as in Peñaranda and Sentana (2014).

We complement our theoretical results with extensive Monte Carlo evidence on the reliability of the usual asymptotic approximations to the finite sample distributions of estimators and tests.

In our empirical application we study the overspecification of the Epstein-Zin model, where the SDF is linear in consumption growth and the market return, using two different cross sections of excess returns: the Fama-French 17 industry sorted portfolios and the 25 book-to-market and size sorted portfolios. We find that the Epstein-Zin model is overspecified when explaining both cross-sections. However, the nature of the overspecification is different for each data set.

When using the 17 portfolios, we find that the model is overspecified due to the presence of a redundant factor. That is, while the market and consumption, separately, can price the returns, together they produce an underidentified model. In contrast, when explaining the returns of the 25 portfolios, the model is identified: we are able to pin down a single (up to scale) valid SDF. However, it is still overspecified, due to the presence of the market return as a factor with no ability to price. An SDF affine in consumption growth would be a valid and more parsimonious model in these data.

The rest of the paper is organized as follows. Section 2 provides the econometric framework for the empirical evaluation of overspecified asset pricing models. We then study empirically several asset pricing models in section 3. We report the results of the simulation evidence in section 4. Finally, we summarize our conclusions and discuss some avenues for further research in section 5. A detailed description of the possible cases with models of one, two, and three factors are relegated to appendix A, while appendix B contains the Monte Carlo design.

2 Overspecified Asset Pricing Models

2.1 Stochastic discount factors and moment conditions

Let $\mathbf{r}$ be an $n \times 1$ vector of excess returns, whose means $E(\mathbf{r})$ we assume are not all equal to zero. Standard arguments such as lack of arbitrage opportunities or the first order conditions of a representative investor imply that

$$E(m\mathbf{r}) = \mathbf{0}$$

for some random variable m called SDF, which discounts uncertain payoffs in such a way that their expected discounted value equals their cost.

The standard approach in empirical finance is to model the SDF as an affine transformation of some $k < n$ observable risk factors $\mathbf{f}$, even though this ignores that m must be positive with probability 1 to avoid arbitrage opportunities, which would require non-linear specifications for m (see Hansen and Jagannathan (1991)). In particular, researchers typically express the pricing equation as

$$E\left[\left(a+\mathbf{b}'\mathbf{f}\right)\mathbf{r}\right]=\mathbf{0} \tag{1}$$

for some real numbers $(a, \mathbf{b})$, which we can refer to as the intercept and slopes of the affine SDF $a + \mathbf{b}'\mathbf{f}$.

When there is a solution different from the trivial one $(a, \mathbf{b}) = (0, \mathbf{0})$, we can at best identify directions in $(a, \mathbf{b})$ space, which leaves both the scale and sign of the SDF undetermined, unless we add an asset whose price is different from 0.[2] As forcefully argued by Hillier (1990) for single equation IV models, this suggests that we should concentrate our efforts in estimating the identified direction. However, empirical researchers often prefer to estimate points rather than directions, and for that reason they typically focus on some asymmetric scale normalization, such as $(1, \mathbf{b}/a)$. In this regard, note that $-\mathbf{b}/a$ can be interpreted as prices of risk since we may rewrite (1) as $E(\mathbf{r}) = E\left(\mathbf{r}\mathbf{f}'\right)(-\mathbf{b}/a)$.

Alternatively, we can express the pricing conditions (1) in terms of central moments. Specifically, we can add and subtract $\mathbf{b}'\boldsymbol{\mu}$ from $a + \mathbf{b}'\mathbf{f}$, define $c = a + \mathbf{b}'\boldsymbol{\mu}$ as the expected value of the affine SDF and re-write the pricing conditions as

$$E\left\{\begin{array}{c}\left[c+\mathbf{b}'\left(\mathbf{f}-\boldsymbol{\mu}\right)\right]\mathbf{r}\\ \mathbf{f}-\boldsymbol{\mu}\end{array}\right\}=\mathbf{0}. \tag{2}$$

The unknown parameters become $(c, \mathbf{b}, \boldsymbol{\mu})$ instead of $(a, \mathbf{b})$, but we have added k extra moments to estimate $\boldsymbol{\mu}$. Once again, we can only identify directions in $(c, \mathbf{b})$ space from (2) under the null hypothesis of existence of some nontrivial solution to (2). In this regard, empirical work usually focuses on $(1, \mathbf{b}/c)$, where $-\mathbf{b}/c$ can also be interpreted as prices of risk because (2) implies that $E(\mathbf{r}) = cov(\mathbf{r}, \mathbf{f})(-\mathbf{b}/c)$.

We refer to those two variants as the uncentred and centred SDF versions since they rely on either $E(\mathbf{r}f)$ or $Cov(\mathbf{r}, f)$ in explaining the cross-section of risk premia. Peñaranda and Sentana (2014) show that the estimators and tests of both variants are numerically equivalent if one uses CU-GMM. In what follows, we will work with the uncentred SDF version, which does

[2] Peñaranda and Sentana (2014) show that the CU-GMM criterion function is numerically invariant to the addition of such an asset. For that reason, we will focus on excess returns in this paper.

not require the estimation of $\boldsymbol{\mu}$, a non-trivial computational advantage.[3] In fact, we can still rely on the uncentred SDF moment conditions (1) to estimate c, which has the interpretation of the SDF mean, as long as we add the moment condition

$$E\left[a+\mathbf{b}'\mathbf{f}-c\right]=0, \tag{3}$$

which is exactly identified for c given $(a, \mathbf{b})$. A non-trivial advantage of this approach is that (1) and (3) are linear in $(a, \mathbf{b}, c)$ while (2) is not linear in $(c, \mathbf{b}, \boldsymbol{\mu})$.

Our empirical applications will consider models where the elements of $\mathbf{f}$ are either nontraded or they are portfolios of $\mathbf{r}$. In those cases, the pricing conditions (1) and (3) contain all the relevant information to estimate and test the asset pricing model. Nevertheless, it would be very easy to extend our analysis to traded factors whose excess returns do not belong to the linear span of $\mathbf{r}$. In that case, we should add to (1) or (2) moment conditions such as

$$E\left[\left(a+\mathbf{b}'\mathbf{f}\right)\mathbf{f}\right]=\mathbf{0}$$

to complete the asset pricing information that we should consider.

2.2 Admissible SDFs sets

The pricing conditions (1) can be expressed in matrix notation as

$$\begin{pmatrix} E(\mathbf{r}) & E(\mathbf{rf}') \end{pmatrix}\begin{pmatrix} a \\ \mathbf{b} \end{pmatrix}=\mathbf{M}\boldsymbol{\theta}=\mathbf{0}, \tag{4}$$

where $\mathbf{M}$ is an $n\times(k+1)$ matrix of data and $\boldsymbol{\theta}$ a $(k+1)\times 1$ parameter vector.

The highest possible rank of $\mathbf{M}$ is its number of columns $k+1$ since we have explicitly assumed that $k<n$. In that case, though, the asset pricing model cannot hold because the only value of $\boldsymbol{\theta}$ that satisfies (4) will be the trivial solution $a=0$, $\mathbf{b}=\mathbf{0}$. On the other hand, if the rank of $\mathbf{M}$ is k then there is a one-dimensional subspace of $\boldsymbol{\theta}$'s that satisfy (4), in which case the solution $\boldsymbol{\theta}$ is unique up to scale, as we have explained in the previous section. Therefore, $rank(\mathbf{M})=k$ coincides with the usual identification condition required for standard GMM inference (see e.g. Hansen (1982) and Newey and McFadden (1994)).

Recently, though, Kan and Zhang (1999) and Burnside (2014) among others have forcefully argued that some empirical asset pricing models effectively rely on factors for which the matrix $Cov(\mathbf{r},\mathbf{f})$ does not have full column rank. The best known example is a useless factor, which would yield a zero column in the matrix $Cov(\mathbf{r},\mathbf{f})$. Alternatively, we may have included two

[3] There is an alternative approach to test asset pricing models, which relies on the regression of $\mathbf{r}$ onto a constant and $\mathbf{f}$. The regression approach requires a higher number of parameters to estimate from a higher number of moments. Nevertheless, the results in Peñaranda and Sentana (2014) show that it provides numerically equivalent tests and prices of risk estimates.

proxies of a relevant pricing factor, so that their difference will be uncorrelated to the vector of excess returns. More generally, there will be rank failures in $Cov(\mathbf{r}, \mathbf{f})$ when we can find a valid asset pricing model with fewer factors. For example, assume that the true model is a (linearized) version of the CCPAM but the factor mimicking portfolio for consumption is perfectly correlated with the market portfolio. In this case both the CAPM and the CCAPM will hold in the sense that the market excess return and consumption growth can price a cross-section of excess returns in a single-factor model. Then an SDF that is linear in both factors, such as the one motivated by the (linearized) Epstein-Zin model, must have a matrix $Cov(\mathbf{r}, \mathbf{f})$ whose rank is one.[4] Given that

$$\begin{pmatrix} E(\mathbf{r}) & E(\mathbf{r}\mathbf{f}') \end{pmatrix} \begin{pmatrix} a \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} E(\mathbf{r}) & Cov(\mathbf{r}, \mathbf{f}) \end{pmatrix} \begin{pmatrix} c \\ \mathbf{b} \end{pmatrix},$$

rank failures in $Cov(\mathbf{r}, \mathbf{f})$ automatically translate into rank failures in $\mathbf{M}$. If the rank of $Cov(\mathbf{r}, \mathbf{f})$ is $k-1$ but the rank of $\mathbf{M}$ is k, the model parameters remain econometric identified, so we can still use standard GMM inference.[5] More generally, though, it is of the upmost importance to develop statistical inference tools that can successfully deal with situations in which $rank(\mathbf{M}) < k$.

Following Arellano, Hansen and Sentana (2012), we begin by specifying the dimension of the subspace of solutions to (4), which we will denote as d, so that the rank of $\mathbf{M}$ will be $(k+1)-d$. Given that we maintain the hypothesis that $E(\mathbf{r}) \neq \mathbf{0}$, we could in principle consider any positive integer d up to a maximum value of k, or equivalently, ranks of $\mathbf{M}$ as low as 1.

As we have mentioned before, when $d = 1$ we can rely on standard GMM to estimate a unique $\boldsymbol{\theta}$ (up to normalization) and use its associated J test to test the validity of the asset pricing restrictions. However, when $d \geq 2$, then we will have a multidimensional subspace of admissible SDFs even after fixing their scale. Nevertheless, we can efficiently estimate a basis of that subspace by replicating d times the moment conditions (4) as follows:

$$\left.\begin{array}{c} \mathbf{M}\boldsymbol{\theta}_1 = \mathbf{0}, \\ \mathbf{M}\boldsymbol{\theta}_2 = \mathbf{0}, \\ \vdots \\ \mathbf{M}\boldsymbol{\theta}_d = \mathbf{0}, \end{array}\right\} \tag{5}$$

and imposing enough normalizations on $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_d)$ to ensure identification of the basis. In this setting, the I test of dimension d is the J test of the extended moment conditions. As for the $c's$, which give the expected values of the basis SDF's, (5) can be supplemented with the

[4] Note that in this case we have three equivalent mean-variance frontiers: The frontier constructed with the cross-section of excess returns, the frontier spanned by the market excess return and the frontier spanned by the consumption mimicking portfolios.

[5] Nevertheless, some asymmetric normalizations may be incompatible with these configurations (see section 4.4 in Peñaranda and Sentana (2014) for further details in the case of a single pricing factor).

moment conditions

$$\left.\begin{array}{c}\begin{pmatrix} 1 & E(\mathbf{f})' \end{pmatrix}\boldsymbol{\theta}_1 - c_1 = 0,\\ \begin{pmatrix} 1 & E(\mathbf{f})' \end{pmatrix}\boldsymbol{\theta}_2 - c_2 = 0,\\ \vdots \\ \begin{pmatrix} 1 & E(\mathbf{f})' \end{pmatrix}\boldsymbol{\theta}_d - c_d = 0,\end{array}\right\} \quad (6)$$

which are exactly identified for given values of $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_d)$.

As for the normalization, the natural asymmetric counterpart to fixing one entry of $\boldsymbol{\theta}$ to 1, which is the usual approach when there is identification, would be to make a $d \times d$ block of (a permutation of) the matrix $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_d)$ equal to the identity matrix of order d, as in Appendix A (see Arellano, Hansen and Sentana (2012) for other symmetric normalizations). Although there will generally be multiple possible blocks, the advantage of single-step methods such as CU-GMM is that inferences will be numerically invariant to the chose normalization.

2.3 Testing restrictions on admissible SDF sets

As we have just seen, our inference framework allows us to estimate the set of SDFs that is compatible with the pricing conditions (1). But we can also use it to test if this subspace satisfies some relevant restrictions. For example, we may want to test if some factor, say f_1, does not appear in any admissible SDF. This test would be associated to the corresponding entry of $\mathbf{b}$ being zero in all the vectors $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_d)$. In some cases, though, such a test will not be interesting because it effectively amounts to $E(\mathbf{r}) = \mathbf{0}$. For instance, when $k = 2$ and $d = 2$ the joint null hypothesis that $b_{11} = b_{21} = 0$ and the asset pricing model is true can only hold if $E(\mathbf{r}) = \mathbf{0}$ and the covariance of f_1 with $\mathbf{r}$ is zero. In contrast, such a degenerate implication would no longer be true if $d = 2$ but $k = 3$ instead.

An important test that we should always add to our analysis is that of zero means for all admissible SDFs, which are associated to the parameters $(c_1, c_2, ..., c_d)$. The reason is that if all these means are zero, then there will be no element in the admissible SDF set that explains the cross-section of expected returns from an meaningful economic perspective. More specifically, when $c_1 = c_2 = ... = c_d = 0$, all the vectors $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_d)$ compatible with the moment restrictions (5) are simply exploiting rank failures in $Cov(\mathbf{r}, \mathbf{f})$, and hence they will be unrelated to the vector $E(\mathbf{r})$.

We can test any of the aforementioned constraints on the set of admissible SDFs by means of distance metric tests, which compare the I statistics computed with and without the constraints imposed on the estimation of the basis.

2.4 Comparison to the previous literature

Burnside (2014) and Gospodinov, Kan and Robotti (2014) apply the rank tests proposed by Cragg and Donald (1997) and Kleibergen and Paap (2006) to $Cov(\mathbf{r}, \mathbf{f})$ and $E(\mathbf{rf}')$, which are the matrices of centred and uncentred second moments between returns and pricing factors, respectively, in order to detect the existence of trivial SDFs that exploit the rank failure in those matrices to account for the cross-section of risk premia in $E(\mathbf{r})$. However, those tests cannot conclude if there are also nontrivial SDFs that can properly explain $E(\mathbf{r})$. In contrast, our econometric framework allows us to estimate a basis of the linear subspace of admissible SDFs, which we can then use to test whether or not all of them are trivial.

The other main difference with Burnside (2014) and Gospodinov, Kan and Robotti (2014) is that they focus their analysis on the implications of those rank failures for standard GMM procedures that assume point identification. In contrast, we develop a GMM framework that works under set identification.

3 Empirical Application

In this empirical application we study the potential overspecification of the Epstein and Zin (1989) model. As is well known, this model effectively assumes that the SDF is linear in consumption growth and the excess return on the market, i.e.:

$$m = a + b_{vw} vw + b_{cg} cg,$$

where vw is the market factor, cg is consumption growth, and c, b_{vw} and b_{cg} are constants to be determined. Note that although the market return is a traded factor, we do not need to add its pricing condition to (1) because it can effectively be generated as a portfolio of the cross-section of excess returns that we want to price.

Assuming that $E(\mathbf{r}) \neq \mathbf{0}$, the model could be overspecified if either $b_{vw} = 0$ or $b_{cg} = 0$. Alternatively, overspecification would also arise if there existed two linearly independent vectors $\boldsymbol{\theta} = (a, b_{vw}, b_{cg})$ and $\boldsymbol{\theta}^* = (a^*, b^*_{vw}, b^*_{cg})$ that would satisfy the following moment conditions:

$$\mathbf{M}\boldsymbol{\theta} = \mathbf{0}$$
$$\mathbf{M}\boldsymbol{\theta}^* = \mathbf{0},$$

where

$$\mathbf{M} = \begin{pmatrix} E(\mathbf{r}) & E(\mathbf{r}vw) & E(\mathbf{r}cg) \end{pmatrix}.$$

If these two vectors exist, then they represent a base of the linear subspace of admissible SDFs, in the sense that linear combinations of them can represent any SDF that can price the securities in $\mathbf{r}$.

We will estimate this model using single-step GMM methods. As discussed before, these methods are invariant to different parametrizations of the SDF, so we will use the uncentred version because it is more parsimonious. In the case of multiple admissible SDF's, the minimization problem that we will solve is:

$$\min_{(\boldsymbol{\theta},\boldsymbol{\theta}^*)} \begin{pmatrix} \hat{\mathbf{M}}\theta \\ \hat{\mathbf{M}}\theta^* \end{pmatrix}' \widehat{\mathbf{V}}^{-1}(\boldsymbol{\theta},\boldsymbol{\theta}^*) \begin{pmatrix} \hat{\mathbf{M}}\theta \\ \hat{\mathbf{M}}\theta^* \end{pmatrix}$$

where

$$\hat{\mathbf{M}} = \begin{pmatrix} \frac{1}{T}\sum_{t=1}^{T}\mathbf{r}_t & \frac{1}{T}\sum_{t=1}^{T}\mathbf{r}_t \cdot vw_t & \frac{1}{T}\sum_{t=1}^{T}\mathbf{r}_t \cdot cg_t \end{pmatrix},$$

$\widehat{\mathbf{V}}(\theta,\theta^*)$ is an estimator of the asymptotic variance of the sample moment conditions that depends on the parameter values. Finally, we will impose suitable normalizations on the vectors $\boldsymbol{\theta}$ and $\boldsymbol{\theta}^*$ in order to point identify a base of the space of admissible SDFs. As we previously mentioned, we augment these moment conditions with the exactly identified moment conditions (6) to estimate the associated c and c^*.

Despite the linearity of the influence function in the model parameters, we rely on numerical optimization to maximize the CU-GMM criterion given its nonlinear nature. Specifically, we compute the CU-GMM criterion by means of the auxiliary OLS regressions described in Peñaranda and Sentana (2012). To reduce the probability of getting trapped in a local minimum, we consider multiple random starting points centred around an inefficient but consistent GMM estimator of $(\boldsymbol{\theta},\boldsymbol{\theta}^*)$ which uses the identity matrix as weighting matrix. For analogous reasons, we perform the optimization using several normalizations of the base vectors, in spite of the fact that CU-GMM is invariant across them.

We use cross-sections of stock returns commonly employed in the literature: the 25 Fama-French portfolios sorted by size and book-to-market, and the 17 Fama-French industry sorted portfolios.[6] Our data spans the period from February 1959 to December 2012 at a monthly frequency. We identify the market factor with the usual Fama and French excess return on the value weighted portfolio, and the consumption factor with the growth rate in real non-durable consumption. Consumption is from the US Department of Commerce, Bureau of Economic Analysis, as in e.g. Lettau and Ludvigson (2001) (see Gospdinov, Kan and Robotti (2014) for further details).

[6] See Ken French web page and Fama and French (1993) for further details.

3.1 Results with 17 Industry Sorted Portfolios

Table 1 presents the empirical results for the 17 Industry sorted portfolios of Fama and French.

(Table 1)

Panel A reports the results of estimating the linearized version of the Epstein-Zin model (1989). As can be seen, the usual overidentification tests does not reject the null hypothesis that there exists an SDF affine in the market excess returns and consumption growth that can price the cross-section of securities (p-value=57.8%). However, the validity of the asymptotic distribution of this J test depends on the model parameters being point identified.

For that reason, in Panel B we report the results of the overidentification tests for $d = 2$ (the I test), which assesses whether there is a subspace of dimension 2 of admissible SDFs that can price the cross section of expected excess returns. Since the p-value of this I test is 13.0%, we take the results as evidence that the model might be overspecified. Given that there are only two factors, there are two possibilities: either one of the factors is useless, in the sense that all the covariances between the excess returns and the factor are 0, or it is redundant given the other factor.

To infer the nature of the overspecification, the estimates of the base shown in Panel B prove useful. The chosen normalization, to fix the scale and ensure linear independence of $\boldsymbol{\theta}$ and $\boldsymbol{\theta}^*$, imposes that each factor, separately, can explain the returns. We take as suggestive evidence that both single-factor SDFs can price the returns the fact that the a's are significantly different from zero for both vectors.

Importantly, though, we must also study if the vector of risk premia lies in the span of the covariance of the returns and the factors. This condition is equivalent to the existence of non-trivial SDFs whose expectation is different from 0 that can price excess returns. In that regard, the estimates of the mean of the two vectors of the base, c and c^*, reported in Panel B, are statistically different from 0 for the reported normalization, providing evidence that there exist admissible SDFs which are not trivial. Additionally, we show the results of the distance metric test of the null hypothesis that $c = 0$ and $c^* = 0$, which is equivalent to all admissible SDFs having 0 mean. Not surprisingly, the null hypothesis is massively rejected by a test statistic of 113.79.[7]

Note that Panel B does not report the distance metric tests of zero price of risk for each one of the factors. As commented in Section 2.3, such a test is not interesting when $k = 2$ and $d = 2$

[7] By using CU-GMM, the distance metric test statistic is numerically invariant to the chosen normalization.

because it effectively amounts to $E(\mathbf{r}) = \mathbf{0}$.

Overall, the evidence provided suggest that we could use a single factor SDF for pricing purposes, and that by adding a second factor, the model becomes both overspecified and underidentified. To confirm this conjecture, we study the pricing properties of the CAPM and the CCAPM on their own, whose SDFs are affine either in the market or in consumption, respectively. The corresponding J tests, which are shown in Panel C of Table 1, show that neither the CAPM nor the CCAPM model is rejected for these data. Moreover, the distance metric tests for the existence of non-trivial SDFs also provide evidence that both SDFs are non-trivial, in the sense that the vector of risk premia is spanned by the covariances of returns and each of the pricing factors, which in turn implies that the covariance of the excess returns and the market and the covariance of the excess returns and consumption are also linearly dependent.

3.2 Results with 25 Size and Book to Market Sorted Portfolios

Table 2 shows the results of the study of overspecification of the Epstein Zin model when pricing the 25 portfolios of Fama and French.

(Table 2)

As in the previous section, Panel A shows the results of the J test, which does not reject. However, Panel B shows that the overidentification tests for $d = 2$ (the I test) clearly rejects the existence of a subspace of dimension 2 of admissible SDFs that can price the cross section of expected excess returns. Thus, there is evidence that the set of admissible SDFs is 1 - dimensional, so that there is only one direction of (a, b_{vw}, b_{cg}) consistent with the vector of expected excess returns.

Once again, we study the existence of a non-trivial SDF with a non-zero mean. The value of c shown in Panel A is significantly different from 0, providing evidence that the risk premia are spanned by the covariance of the returns and the factors. This is confirmed by the distance metric test of the null hypothesis of $c = 0$, which is marginally rejected with a p-value of 4.7%.

Despite these positive results, the Epstein Zin model would still be overspecified if we could find an admissible SDF with only one factor. In this regard, the results reported in Panel A show that the coefficient associated with the market is not significantly different from zero, while the coefficient associated with consumption is. These Wald-type statistics are supported by distance metric tests that impose the null hypothesis that the price of risk associated to the market or consumption growth are 0 (3.033 with a p-value of 8.2% for the market and 75.897 with a negligible p-value for consumption). Therefore, our empirical evidence suggests that although the covariances of the 25 Fama and French portfolios with the value weighted portfolio are not proportional to the vector of expected excess returns, the inclusion of consumption growth in the SDF ensures that the model is not rejected in this data.

Finally, we look at the behavior of the CAPM and the CCAPM separately. Not surprisingly, the CAPM is rejected in these data while the CCAPM is not. Moreover in the case of the CCAPM the distance metric test of a trivial SDF is also rejected with a p-value of 1.48%. Nevertheless, it is important to emphasize that the price of risk associated to consumption is very high, as expected from the equity premium puzzle.

Overall, we find that the Epstein-Zin model is not rejected in this data, and that the SDF is well identified. However, it contains a risk factor, i.e. the market, that has no pricing power, so that an SDF affine in only consumption growth seems able to explain the cross-section of expected excess returns.

4 Monte Carlo (to be completed)

The design of our Monte Carlo experiment is described in Appendix B.

5 Conclusions

We study the testing of linear factor pricing models and the estimation of risk prices in potentially overspecified contexts in which we can only estimate the set of risk prices compatible with the pricing conditions. We use single-step GMM procedures, such as continuously updated GMM, to obtain identical test statistics and risk price estimates for SDF and regression methods, with uncentred or centred moments and symmetric or asymmetric normalizations. We also develop simple tests that can detect problematic cases such as the presence of trivial SDFs unrelated to the cross-section of returns. We provide extensive Monte Carlo evidence on estimators and tests.

In our empirical application we study the overspecification of the Epstein-Zin model, where the SDF is linear in consumption growth and the market return, using two different cross sections of excess returns: the Fama-French 17 industry sorted portfolios and the 25 book-to-market and size sorted portfolios. We find that the Epstein-Zin model is overspecified when explaining both the 17 portfolios and the 25 portfolios of excess returns. However, the nature of the overspecification is different for each cross section.

When using the 17 portfolios, we find that the model is overspecified due to the presence of a redundant factor. That is, while the market and consumption, separately, can price the returns, together they produce an underidentified model. In contrast, when explaining the returns of the 25 portfolios, the model is identified: we are able to pin down a single (up to scale) valid SDF.

However, it is still overspecified, due to the presence of the market return as a factor with no ability to price. An SDF affine in consumption growth would be a valid and more parsimonious model in these data.

References

Almeida, C. and R. Garcia (2012): "Assessing misspecified asset pricing models with empirical likelihood estimators", *Journal of Econometrics* 170, 519–537.

Arellano, M., L.P. Hansen and E. Sentana (2012): "Underidentification?", *Journal of Econometrics* 170, 256–280.

Burnside, C. (2014): "Identification and inference in linear stochastic discount factor models", mimeo, Duke.

Cochrane, J.H. (2001): *Asset Pricing*, Princeton University Press.

Cragg, J.G. and S.G. Donald (1997): "Inferring the rank of a matrix", *Journal of Econometrics* 76, 223–250.

Epstein, L.G. and S.E. Zin (1989): "Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework", *Journal of Political Economy* 57, 937–969.

Fama, E.F. and K.R. French (1993): "Common risk factors in the returns on stock and bonds", *Journal of Financial Economics* 33, 3–56.

Gospodinov, N., R. Kan, and C. Robotti (2014): "Spurious fit in unidentified asset-pricing models", mimeo, Federal Reserve Bank of Atlanta.

Hansen, L.P. (1982): "Large sample properties of generalized method of moments estimators", *Econometrica* 50, 1029–1054.

Hansen, L.P., J. Heaton and A. Yaron (1996): "Finite sample properties of some alternative GMM estimators", *Journal of Business and Economic Statistics* 14, 262–280.

Hansen, L.P. and R. Jagannathan (1991): "Implications of security market data for models of dynamic economies", *Journal of Political Economy* 99, 225–262.

Hansen, L.P. and R. Jagannathan (1997): "Assessing specification errors in stochastic discount factor models", *Journal of Finance* 52, 557–590.

Harvey, C. R., Y. Liu, and H. Zhu (2014): "... and the cross-section of expected returns", mimeo, Fuqua Business School, Duke University.

Hillier, G.H. (1990): "On the normalization of structural equations: properties of direct estimators", *Econometrica* 58, 1181–1194.

Householder, A.S. (1964): *The Theory of Matrices in Numerical Analysis*, Blaisdell Publishing Co., London.

Jagannathan, R. and Z. Wang (1996): "The conditional CAPM and the cross-section of expected returns", *Journal of Finance* 51, 3–53.

Kan, R. and C. Robotti (2008): “Specification tests of asset pricing models using excess returns”, *Journal of Empirical Finance* 15, 816–38.

Kan, R. and C. Zhang (1999): “GMM tests of stochastic discount factor models with useless factors”, *Journal of Financial Economics* 54, 103–127.

Kleibergen, F. and R. Paap (2006): “Generalized reduced rank tests using the singular value decomposition”, *Journal of Econometrics* 133, 97–126.

Lettau, M., and S. Ludvigson (2001): “Resurrecting the (C)CAPM: a cross-sectional test when risk premia are time-varying”, *Journal of Political Economy* 109, 1238–1287.

Lewellen J., S. Nagel and J. Shanken (2010): “A skeptical appraisal of asset-pricing tests, *Journal of Financial Economics* 96, 175–194.

Manresa, E. (2009): “Application of rank tests to linear factor asset pricing models”, unpublished Master thesis, CEMFI.

Newey, W.K. and D.L. McFadden (1994): “Large sample estimation and hypothesis testing”, in Engle, R.F. and D.L. McFadden, *Handbook of Econometrics* vol. IV, 2111–2245, Elsevier.

Newey, W.K. and R.J. Smith (2004): “Higher order properties of GMM and generalized empirical likelihood estimators”, *Econometrica* 72, 219–255.

Peñaranda, F. and E. Sentana (2012): “Spanning tests in portfolio and stochastic discount factor mean-variance frontiers: a unifying approach”, *Journal of Econometrics* 170, 303–324.

Peñaranda, F. and E. Sentana (2014): “A unifying approach to the empirical evaluation of asset pricing models”, forthcoming in *Review of Economics and Statistics.*

Sargan, J.D. (1958): “The estimation of economic relationships using instrumental variables”, *Econometrica* 26, 393–415.

Yogo, M. (2006): “A consumption-based explanation of expected stock returns”, *Journal of Finance* 61, 539–580.

Appendices

A Possible cases with one, two and three factors

Our empirical applications consider models where the elements of $\mathbf{f}$ are either nontraded or they are portfolios of $\mathbf{r}$. In those cases, the pricing conditions (1) or, equivalently, the matrix $\mathbf{M}$ in (4) contain all the relevant information to estimate and test the asset pricing model.

We describe all the possible cases for models with one, two or three factors under the only maintained assumption that $E(\mathbf{r}) \neq \mathbf{0}$. This assumption implies that we only need to study cases where the rank of $\mathbf{M}$ is one or higher. We also describe some normalizations for each case that we can use to implement GMM.

A.1 One factor

We cannot have an underidentified single-factor model because the valid SDFs are unique up to scale:

- Identification: The rank of $\mathbf{M}$ is one.
  - $E(\mathbf{r})$ is not in the span of $Cov(\mathbf{r}, f)$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is zero.
  - $E(\mathbf{r})$ is in the span of $Cov(\mathbf{r}, f)$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is one.
- Lack of a valid SDF: The rank of $\mathbf{M}$ is one, while the rank of $Cov(\mathbf{r}, \mathbf{f})$ is one.

In a single-factor model, the parameter vector that we estimate is

$$\boldsymbol{\theta} = \begin{pmatrix} a \\ b \end{pmatrix}$$

subject to a normalization. For instance, if we normalize one entry to one then we have two possible normalizations

$$\begin{pmatrix} 1 \\ b \end{pmatrix}, \begin{pmatrix} a \\ 1 \end{pmatrix}.$$

A.2 Two factors

The valid SDFs may belong to a two-dimensional subspace and hence we may find underidentified two-factor models:

- Underidentification of dimension two: The rank of $\mathbf{M}$ is one.
  - $E(\mathbf{r})$ is not in the span of $Cov(\mathbf{r}, \mathbf{f})$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is zero.

- $E(\mathbf{r})$ is in the span of $Cov(\mathbf{r}, \mathbf{f})$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ one.

- Identification: The rank of $\mathbf{M}$ is two.
  - $E(\mathbf{r})$ is not in the span of $Cov(\mathbf{r}, \mathbf{f})$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is one.
  - $E(\mathbf{r})$ is in the span of $Cov(\mathbf{r}, \mathbf{f})$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is two.
- Lack of a valid SDF: The rank of $\mathbf{M}$ is three, while the rank of $Cov(\mathbf{r}, \mathbf{f})$ is two.

In a two-factor model, when $d = 1$, the parameter vector that we estimate is

$$\boldsymbol{\theta} = \begin{pmatrix} a \\ b_1 \\ b_2 \end{pmatrix}$$

subject to a normalization. If we want to normalize one entry to one then we have three possible normalizations

$$\begin{pmatrix} 1 \\ b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} a \\ 1 \\ b_2 \end{pmatrix}, \begin{pmatrix} a \\ b_1 \\ 1 \end{pmatrix}.$$

If we analyze the case $d = 2$ instead, then we estimate two parameter vectors

$$(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \begin{pmatrix} a_1 & a_2 \\ b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

subject to normalizations. If we use the identity matrix to normalize these vectors then we have three possible normalizations

$$\begin{pmatrix} a_1 & a_2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ b_{11} & b_{12} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ b_{21} & b_{22} \end{pmatrix}.$$

A.3 Three factors

The valid SDFs may belong to a three-dimensional subspace:

- Underidentification of dimension three: The rank of $\mathbf{M}$ is one.
  - $E(\mathbf{r})$ is not in the span of $Cov(\mathbf{r}, \mathbf{f})$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is zero.
  - $E(\mathbf{r})$ is in the span of $Cov(\mathbf{r}, \mathbf{f})$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is one.

- Underidentification of dimension two: The rank of $\mathbf{M}$ is two.
  - $E(\mathbf{r})$ is not in the span of $Cov(\mathbf{r}, \mathbf{f})$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is one.
  - $E(\mathbf{r})$ is in the span of $Cov(\mathbf{r}, \mathbf{f})$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is two.
- Identification: The rank of $\mathbf{M}$ is three.
  - $E(\mathbf{r})$ is not in the span of $Cov(\mathbf{r}, \mathbf{f})$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is two.
  - $E(\mathbf{r})$ is in the span of $Cov(\mathbf{r}, \mathbf{f})$: The rank of $Cov(\mathbf{r}, \mathbf{f})$ is three.
- Lack of a valid SDF: The rank of $\mathbf{M}$ is four, while the rank of $Cov(\mathbf{r}, \mathbf{f})$ is three.

In a three-factor model, when $d = 1$, the parameter vector that we estimate is

$$\boldsymbol{\theta} = \begin{pmatrix} a \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

subject to a normalization. There are four possible normalizations if we fix one entry to one

$$\begin{pmatrix} 1 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}, \begin{pmatrix} a \\ 1 \\ b_2 \\ b_3 \end{pmatrix}, \begin{pmatrix} a \\ b_1 \\ 1 \\ b_3 \end{pmatrix}, \begin{pmatrix} a \\ b_1 \\ b_2 \\ 1 \end{pmatrix}.$$

If we analyze the case $d = 2$, then we estimate two parameter vectors

$$(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \begin{pmatrix} a_1 & a_2 \\ b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

subject to normalizations. There are six possible normalizations if we use the identity matrix

$$\begin{pmatrix} a_1 & a_2 \\ b_{11} & b_{12} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 \\ 1 & 0 \\ b_{21} & b_{22} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 \\ 1 & 0 \\ 0 & 1 \\ b_{31} & b_{32} \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ b_{11} & b_{12} \\ b_{21} & b_{22} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ b_{11} & b_{12} \\ 0 & 1 \\ b_{31} & b_{32} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}.$$

Finally, if we analyze the case $d = 3$, then we estimate three parameter vectors

$$(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \boldsymbol{\theta}_3) = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

subject to normalizations. There are four possible normalizations if we use the identity matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

B Monte Carlo design

We develop a Monte Carlo experiment with a two-factor model, $k = 2$, in a similar spirit to the one-factor design in Peñaranda and Sentana (2014). An unrestricted Gaussian data generating process (DGP) for $(\mathbf{f}, \mathbf{r})$ is

$$\mathbf{f} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}),$$
$$\mathbf{r} = \boldsymbol{\mu}_r + \mathbf{B}_r(\mathbf{f} - \boldsymbol{\mu}) + \mathbf{u}_r, \qquad \mathbf{u}_r \sim N(\mathbf{0}, \boldsymbol{\Omega}_{rr}),$$

where the $n \times 2$ matrix $\mathbf{B}_r$ is defined by the two beta vectors

$$\mathbf{B}_r = \begin{pmatrix} \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 \end{pmatrix}.$$

However, given that we use the simulated data to test that an affine function of $\mathbf{f}$ is orthogonal to $\mathbf{r}$, the only thing that matters is the linear span of $\mathbf{r}$. As a result, we can substantially reduce the number of parameters characterizing the conditional DGP for $\mathbf{r}$ by means of the following steps:

1. a Cholesky transformation of $\mathbf{r}$ to get a residual variance $\boldsymbol{\Omega}_{rr}$ equal to the identity matrix,
2. a Householder transformation that makes the second to the last entries of the vector of risk premia $\boldsymbol{\mu}_r$ equal to zero (see Householder (1964)),

3. another Householder transformation that makes the third to the last entries of $\boldsymbol{\beta}_1$ equal to zero, and

4. A final third Householder transformation that makes the fourth to the last entries of $\boldsymbol{\beta}_2$ equal to zero.

We also construct the two factors such that their variance is the identity matrix. As a result, our simplified DGP for excess returns will be

$$\mathbf{r} = \mu_r \mathbf{e}_1 + (\beta_{11}\mathbf{e}_1 + \beta_{21}\mathbf{e}_2)(f_1 - \mu_1) + (\beta_{21}\mathbf{e}_1 + \beta_{22}\mathbf{e}_2 + \beta_{23}\mathbf{e}_3)(f_2 - \mu_2) + \mathbf{u}_r,$$

$$\mathbf{u}_r \sim N(\mathbf{0}, \mathbf{I}_n),$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are the first, second and third columns of the identity matrix, and

$$\mathbf{f} \sim N(\boldsymbol{\mu}, \mathbf{I}_2).$$

The two parameters in $\boldsymbol{\mu}$ can be directly calibrated from data on $\mathbf{f}$. In turn, the six parameters that define $\mathbf{r}$ can be calibrated as follows. We can define a Hansen-Jagannathan (HJ) distance for this two-factor model as the minimum with respect to $\boldsymbol{\phi}$ of the quadratic form

$$\boldsymbol{\phi}'\mathbb{M}'Var^{-1}(\mathbf{r})\mathbb{M}\boldsymbol{\phi},$$

where

$$\mathbb{M}\boldsymbol{\phi} = \begin{pmatrix} E(\mathbf{r}) & Cov(\mathbf{r}, \mathbf{f}) \end{pmatrix} \begin{pmatrix} c \\ \mathbf{b} \end{pmatrix}.$$

Note that $\mathbb{M}\boldsymbol{\phi} = \mathbf{M}\boldsymbol{\theta}$ and the rank of $\mathbb{M}$ is equal to the rank of $\mathbf{M}$.

The 3×3 weighting matrix

$$\mathbb{M}'Var^{-1}(\mathbf{r})\mathbb{M} = \begin{pmatrix} E(\mathbf{r})'Var^{-1}(\mathbf{r})E(\mathbf{r}) & E(\mathbf{r})'Var^{-1}(\mathbf{r})Cov(\mathbf{r}, \mathbf{f}) \\ \cdot & Cov(\mathbf{r}, \mathbf{f})'Var^{-1}(\mathbf{r})Cov(\mathbf{r}, \mathbf{f}) \end{pmatrix}$$

can be interpreted as the variance matrix of three important portfolios, one that yields the maximum Sharpe ratio

$$r_0 = \mathbf{r}'Var^{-1}(\mathbf{r})E(\mathbf{r}),$$

and the two factor mimicking portfolios

$$r_1 = \mathbf{r}'Var^{-1}(\mathbf{r})Cov(\mathbf{r}, f_1), \qquad r_2 = \mathbf{r}'Var^{-1}(\mathbf{r})Cov(\mathbf{r}, f_2).$$

Note that if we minimize the quadratic form subject to the symmetric normalization $\boldsymbol{\phi}'\boldsymbol{\phi} = 1$ then this HJ distance is equal to the minimum eigenvalue of this variance matrix.

The first entry of the weighting matrix is the variance of r_0 or, equivalently, the squared maximum Sharpe ratio. The other two diagonal entries are the variances of r_1 and r_2 or, equivalently, the R^2 of their respective regressions. Finally, the three different off-diagonal elements correspond to the covariances between these three portfolios, which we can pin down by their correlations. In this way, we have six parameters that are easy to interpret and calibrate, and from them we can pin down the six parameters that our DGP requires for $\mathbf{r}$.

Moreover, it is easy to impose all the scenarios that we are interested in. All the cases have the same $\boldsymbol{\mu}$ They also share the same value of the maximum Sharpe ratio. This value pins down μ_r given the rest of parameter values.

- Underidentification of dimension two when $E(\mathbf{r})$ is not in the span of $Cov(\mathbf{r}, \mathbf{f})$: In this case the rank of $Cov(\mathbf{r}, \mathbf{f})$ is zero but the rank of $\mathbb{M}$ is one. This case corresponds to a matrix $\mathbb{M}'Var^{-1}(\mathbf{r})\mathbb{M}$ where only the first entry is different from zero. The five betas in our DGP are equal to zero.

- Underidentification of dimension two when $E(\mathbf{r})$ is in the span of $Cov(\mathbf{r}, \mathbf{f})$: In this case the rank of both $Cov(\mathbf{r}, \mathbf{f})$ and $\mathbb{M}$ is one. This case corresponds to a matrix $\mathbb{M}'Var^{-1}(\mathbf{r})\mathbb{M}$ where only the first 2×2 block is different from zero if we focus on the extreme case of a useless f_2. Nevertheless, this first block must be singular. All the betas in our DGP are equal to zero except β_{11}. We chose its value to match a particular value of R^2 in the regression that defines r_1.

- Identification when $E(\mathbf{r})$ is not in the span of $Cov(\mathbf{r}, \mathbf{f})$: In this case the rank of $Cov(\mathbf{r}, \mathbf{f})$ is one but the rank of $\mathbb{M}$ is two. This case corresponds to a matrix $\mathbb{M}'Var^{-1}(\mathbf{r})\mathbb{M}$ where only the first 2×2 block is different from zero if we focus once again on the extreme case of a useless f_2. This first block is nonsingular in this case. Both β_{11} and β_{12} are different from zero, while we keep the other three betas at zero. Now we add the information of a particular correlation between r_0 and r_1.

- Identification when $E(\mathbf{r})$ is in the span of $Cov(\mathbf{r}, \mathbf{f})$: In this case the rank of both $Cov(\mathbf{r}, \mathbf{f})$ and $\mathbb{M}$ is two. This case corresponds to a matrix $\mathbb{M}'Var^{-1}(\mathbf{r})\mathbb{M}$ with a single zero eigenvalue again. The novelty is that this singularity is not due to a useless f_2, but to r_0 being spanned by (r_1, r_2). Our DGP satisfies this property by choosing $\beta_{23} = 0$ and a full rank matrix
$$\begin{pmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \end{pmatrix}.$$

Now we add the information of a given value of R^2 in the regression that defines r_2 and a given correlation between r_0 and r_2. The correlation between r_1 and r_2 is implicitly determined by the other two correlations since we must have a singularity.

- Lack of a valid SDF: In this case the rank of $Cov(\mathbf{r}, \mathbf{f})$ is two but the rank of $\mathbb{M}$ is three. This case corresponds to parameters such that the matrix $\mathbb{M}'Var^{-1}(\mathbf{r})\mathbb{M}$ has full rank and hence the HJ distance cannot be zero. The DGP parameters satisfy $\beta_{23} \neq 0$ and full rank of the matrix

$$\begin{pmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \end{pmatrix}.$$

  Now we add the information of a particular correlation between r_1 and r_2, different from the implicit one in the previous case.

Table 1:
Empirical evaluation of the (linearized) Epstein-Zin model with 17 industry sorted portfolios

Panel A. One-dimensional set of SDFs

a	1	I_0	I_0-I
b_{vw}	0.378	13.373	0.052
	(1.525)	(0.645)	(0.820)
b_{cg}	-114.664	19.905	6.584
	(37.603)	(0.225)	(0.010)
c	0.849	21.160	7.839
	(0.055)	(0.172)	(0.005)
J test	13.321		
	(0.058)		

Panel B. Two-dimensional set of SDFs

a	-0.393	-0.009	I_0	I_0-I
	(0.130)	(0.003)		
b_{vw}	1	0		
b_{cg}	0	1		
c	-0.389	-0.008	232.4	191.304
	(0.131)	(0.003)	(0.000)	(0.000)
I test	41.095			
	(0.130)			

Panel C. One-dimensional set of SDFs for single-factor models

	CAPM			CCAPM		
a	1	I_0	I_0-I	1	I_0	I_0-I
b_{vw}	-2.747	28.943	9.038			
	(0.926)	(0.035)	(0.003)			
b_{cg}				-107.537	28.943	15.570
				(27.136)	(0.035)	(0.000)
c	0.986	201.128	181.223	0.857	30.508	17.135
	(0.009)	(0.000)	(0.000)	(0.047)	(0.023)	(0.000)
I test	19.905			13.373		
	(0.225)			(0.645)		

Note: This table displays estimates of the SDF parameters with standard errors in parenthesis, as well as the I tests with p-values in parenthesis. Panel A and B report the results for sets of SDFs of dimension 1 and 2, respectively. Panel C reports the results for the CAPM and the CCAPM. We display the estimates for a particular normalization, but our results are numerically invariant to the chosen normalization because we implement each setting by continuously updated GMM. The I tests are complemented with significance tests of some SDF parameters. In particular, the I_0 test is the I test that imposes the null hypothesis of a zero parameter and the I_0-I test is the distance metric test of that hypothesis. The payoffs to price are monthly excess returns from February 1959 to December 2012.

Table 2:
Empirical evaluation of the (linearized) Epstein-Zin model with 25 size and BM sorted portfolios

Panel A. One-dimensional set of SDFs

a	1	I_0	I_0-I
b_{vw}	4.620	33.089	3.033
	(2.979)	(0.102)	(0.082)
b_{cg}	-374.900	105.953	75.897
	-101.526	(0)	(0)
c	0.535	33.984	3.928
	(0.132)	(0.085)	(0.047)
J test	30.056		
	(0.148)		

Panel B. Two-dimensional set of SDFs

a	-0.224	-0.005	I_0	I_0-I
	(0.048)	(0.001)		
b_{vw}	1	0		
b_{cg}	0	1		
c	-0.216	-0.004	304.555	148.681
	(0.049)	(0.001)	(0.000)	(0.000)
I test	155.874			
	(0)			

Panel C. One-dimensional set of SDFs for single-factor models

	CAPM			CCAPM		
a	1	I_0	I_0-I	1	I_0	I_0-I
b_{vw}	-3.539	114.171	8.218			
	(1.003)	(0.000)	(0.004)			
b_{cg}				-325.424	114.171	81.082
				(76.166)	(0.000)	(0.000)
c	0.976	253.974	148.021	0.565	39.026	5.937
	(0.012)	(0.000)	(0.000)	(0.112)	(0.037)	(0.015)
I test	105.953			33.089		
	(0.000)			(0.102)		

Note: This table displays estimates of the SDF parameters with standard errors in parenthesis, as well as the I tests with p-values in parenthesis. Panel A and B report the results for sets of SDFs of dimension 1 and 2, respectively. Panel C reports the results for the CAPM and the CCAPM. We display the estimates for a particular normalization, but our results are numerically invariant to the chosen normalization because we implement each setting by continuously updated GMM. The I tests are complemented with significance tests of some SDF parameters. In particular, the I_0 test is the I test that imposes the null hypothesis of a zero parameter and the I_0-I test is the distance metric test of that hypothesis. The payoffs to price are monthly excess returns from February 1959 to December 2012.